University of Tehran Press

Interdisciplinary Journal of Management Studies (IJMS) Online ISSN: 2981-0795

Home Page: https://ijms.ut.ac.ir

New Multiplicative Models to Evaluate and Merge Units in Data Envelopment Analysis

Aliraza Fakharzadeh Jahromi[1*] | Hassan Rostamzadeh[2]

1. Corresponding Author, Department of Mathematics, Shiraz University of Technology, Shiraz, Iran. E-mail: a_fakharzadeh@sutech.ac.ir
2. Department of Mathematics, Shiraz University of Technology, Shiraz, Iran. E-mail: h.rostamzadeh@sutech.ac.ir

ARTICLE INFO

Article type:
Research Article

Article History:
Received: 20 July 2024
Revised: 24 November 2025
Accepted: 10 May 2026
Published Online: 01 November 2026

Keywords:
Evaluation,
Merger,
Pareto-efficiency,
Multiplicative model.

ABSTRACT

This study presents new multiplicative models to evaluate and merge decision-making units in the production possibilities set with convex geometry and variable returns to scale technology. The first proposed model evaluates the decision-making units as non-radial and input and output-oriented and accurately identifies all Pareto-efficient and inefficient units. Then, in the second model, a new unit is formed by merging units with optimal inputs and outputs, based on predetermined efficiency targets; in this regard, the contribution of each input and output from each merged unit is determined, and it is shown that the contribution ratio of each input and output is the same in all units. By using the new non-radial model, managers can manage the efficiency of all inputs and outputs of the newly obtained unit in the merger procedure. In the numerical section, the proposed models are employed to evaluate and merge banks in Iran.

Cite this article: Fakharzadeh Jahromi, A. & Rostamzadeh, H. (2027). New Multiplicative Models to Evaluate and Merge Units in Data Envelopment Analysis. *Interdisciplinary Journal of Management Studies* (*IJMS*), 20 (1), 21-42. http//doi.org/10.22059/ijms.2026.379907.676905

© Authors retain the copyright and full publishing rights.
Publisher: University of Tehran Press.
DOI: http//doi.org/10.22059/ijms.2026.379907.676905

1. Introduction

In addition to accurately evaluating an organization's units, one of the essential means of enhancing profitability, reducing costs, and achieving growth in organizations is the merger of two or more decision-making units (DMUs) (Cherian & Jarrow, 1995; Gaughan, 2013). For this reason, management is continually seeking merger models that not only optimize resource utilization and increase output but also improve the organization's efficiency during the merger process. Accordingly, the data envelopment analysis (DEA) has discussed the merger issue. DEA can assess the performance of merged units and highlight areas with potential for improvement (Xie et al., 2019); therefore, it can create an efficient new unit by determining the optimal contribution of resources and production from each merged unit. In mergers, it is commonly assumed that the inputs and outputs of the units can be aggregated by simple addition, but in many cases, this approach is not effective. For instance, Cooper et al. (2007) showed that under conventional DEA frameworks, including the model introduced by Banker et al. (1984), the unit resulting from a merger does not necessarily achieve efficiency. In some cases, even the combination of two efficient units may fail to produce an efficient outcome (Tone, 1996). Therefore, a merger is considered successful only when the resulting unit attains the anticipated efficiency level.

Generally, researchers agree that merger activities contribute to higher profitability and greater efficiency (Hannan & Pilloff, 2009). Healy et al. (1992) found that post-merger improvements in operating cash flows are closely linked to pre-merger abnormal stock returns. Using standard DEA frameworks, Bogetoft and Wang (2005) conducted the merger at the level of available inputs and outputs, calculating the potential profit of the merged entities by optimizing the reduction of inputs. Al Sharkas et al. (2008) conducted a comprehensive empirical study on the cost and efficiency impacts of unit mergers, employing both a parametric methodology and DEA for the analysis. Lozano and Villa (2010) introduced two DEA frameworks; one designed to reduce input costs after a merger, and the other to enhance post-merger profits. In a merger, a critical challenge is determining the input and output levels for each merged DMU to ensure that the resulting unit attains the desired efficiency target. Gattoufi et al. (2014) introduced a novel inverse data envelopment analysis (Inv DEA) (Amin & Emrouznejad, 2007; Emrouznejad et al., 2019; Kalantary et al., 2019; Pendharkar, 2002; Wei et al., 2000) for merging units. In their approach, a method based on Inv DEA was developed to determine the necessary input and output levels needed to attain a specified efficiency target. Using the Inv DEA method, Amin et al. (2017a) developed a model to determine new units with target efficiency levels by combining multiple units. Amin et al. (2017b) also introduced an Inv DEA-based model to forecast the effects of a merger on market outcomes. Subsequently, Amin and Boamah (2020) employed an Inv DEA model based on cost efficiency to predict the potential profits resulting from the merger. In practical applications, Pyra and Siedlecka (2022) used a DEA-based simulation approach to assess the potential outcomes of merging Polish universities. Lozano and Contreras (2022) examined the Spanish public university system to maximize operational efficiency and assess the possible benefits of regional university mergers. In 2022, Soltanifar et al. (2024) introduced a model to handle the merger of units with negative data. Also, the authors merged units using the common set of weights (CSW) based on goal programming. Recently, Fakharzadeh and Rostamzadeh (2025) proposed a model for merging units aimed at obtaining a Pareto-efficient unit operating at the most productive scale size.

Additionally, the merger issue has been examined using the network DEA framework. For example, Xiao et al. (2017) introduced a two-stage network DEA model to assess and analyze the potential cost savings from mergers. Additionally, Khoveyni and Eslami (2022) applied the merger framework to units structured in a two-stage series network using Inv DEA.

From another perspective, the production possibility set (PPS) represents the collection of all input-output combinations that a system can achieve, assuming the convexity property holds. The surface of PPS (border) is called the efficient frontier or the production frontier; in PPS with the technology of convex arithmetic, the efficient frontier is a union of segments, but with the technology of convex geometry, it is a non-concave curve (Cooper et al., 2007). DEA models with convex arithmetic technology are unable to accurately measure the efficiency of DMUs located on the efficient frontier (Kao, 2023). DEA models employing convex geometry technology, also known as multiplicative models, measure the efficiency of DMUs more accurately than non-multiplicative models (Kao, 2023).

This paper presents two new multiplicative models to evaluate and merge DMUs in PPS with the technology of convex geometry and the variable returns to scale property. The first proposed model can evaluate units as non-radial and identify Pareto-efficient and all inefficient units. In contrast, radial multiplicative models evaluate units radially and identify only weakly efficient and purely technically inefficient units. Then, using the first model, a multiplicative model is constructed to merge DMUs; in the presented model, the merger is done with the predetermined efficiency target for the new unit. In this regard, in the new unit, the contribution of each input and output from each merged unit is determined, where the contribution ratio of each input and output is not zero, and is the same in all units. However, in the previously presented methods based on convex technology, the contribution of some units to the merger may be zero, and the contribution ratio of each input and output is not the same across all units.

Another gap in the previous merger models is that a new unit with the optimal inputs or outputs is separately obtained. In contrast, the newly presented model obtains a new unit with the optimal inputs and outputs simultaneously. Additionally, managers using the presented non-radial model can manage the efficiency of all indicators of the obtained new unit from the merger (input and output efficiency refers to the amount of input consumption and output production that a unit can be converted into a Pareto-optimal unit (Cooper et al., 2007)). Moreover, they obtain a new unit with a non-radial efficiency score. In contrast, in previous studies, managers could control the efficiency of the newly obtained merged unit only through a radial approach. The technology of previous models of merger is all convex arithmetic, and according to the available literature, a model with convex geometry technology has not been presented yet.

The structure of this paper is as follows: In Section 2, the required concepts of DEA are reviewed and explained. Section 3 introduces a new multiplicative model and its properties for evaluating DMUs. After a short review of the merger process and relevant DEA models, Section 4 introduces a novel multiplicative model for merging decision-making units, and its properties and advantages are presented. Section 5 compares the presented models with some previous models. To demonstrate the practicality and managerial relevance of the proposed approaches, a real-world case study is presented in Section 6. Then, a general discussion of the features of the presented models is provided in Section 7. The final section is devoted to some conclusions and remarks.

2. Some Required Concepts of DEA

Several DEA models with different technologies have been proposed to evaluate DMUs (Banker et al., 1984; Bolós et al., 2023; Charnes et al., 1978; Cooper et al. 2007; Fang et al., 2024). Charnes et al. (1978), based on Farrell's work in Farrell (1957), presented the CCR model with the technology of convex arithmetic and constant returns to scale ($PPS_{(CRS)}$), which is the base model in DEA. After that, other models, such as BCC (Banker et al., 1984), were introduced with the technology of convex arithmetic and variable returns to scale ($PPS_{(VRS)} = \{(X,Y) | \sum_{j=1}^{n} \lambda_j X_j \leqslant X, \sum_{j=1}^{n} \lambda_j Y_j \geqslant Y, 1\lambda = 1, \lambda \geqslant 0\}$). In the sequel, researchers introduced other DEA models with convex geometric technology (Charnes et al., 1982; Cooper et al., 2007). PPS, with the technology of convex geometry and variable returns to scale, is $PPS_{(GVRS)} = \{(X,Y) | \prod_{j=1}^{n} X_j^{\lambda_j} \leqslant X, \prod_{j=1}^{n} Y_j^{\lambda_j} \geqslant Y, 1\lambda = 1, \lambda \geqslant 0\}$; the efficient frontier of $PPS_{(GVRS)}$ is a non-concave curve (Cooper et al., 2007). Units in $PPS_{(GVRS)}$ are evaluated using multiplicative models, which, compared to non-multiplicative models, provide a more accurate measurement of DMUs efficiency (Kao, 2023).

Definition 2.1

Suppose that, for $j = 1, 2, \ldots, n$, $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj}) \geqslant 1$ and $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj}) \geqslant 1$ are the input and output vectors of $DMU_j \in PPS_{(GVRS)}$. Then, the radial input-oriented multiplicative model for evaluating $DMU_o \in PPS_{(GVRS)}$ is defined as follows (Cooper et al., 2007):

$$
\begin{aligned}
&Min \quad \theta \\
&S.to: \quad \prod_{j=1}^{n} x_{ij}^{\lambda_j} \leqslant x_{io}^{\theta}, \quad i=1,2,...,m; \\
&\qquad \prod_{j=1}^{n} y_{rj}^{\lambda_j} \geqslant y_{ro}, \quad r=1,2,...,s; \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1; \\
&\qquad \lambda_j \geqslant 0, \quad j=1,2,...,n.
\end{aligned} \tag{1}
$$

Definition 2.2
A unit (X, Y) is considered strongly efficient, or Pareto-Koopmans efficient, if there does not exist any $(X, Y) \in PPS$, such that $\bar{X} \leqslant X, \bar{Y} \geqslant Y$, and $(\bar{X}, \bar{Y}) \neq (X, Y)$.

Definition 2.3.
A unit (X, Y) is considered weakly efficient, or Farrell-efficient, if there does not exist any $(\bar{X}, \bar{Y}) \in PPS$, such that $\bar{X} < X$ and $\bar{Y} > Y$.

In evaluating DMU_o using the multiplicative model (1), if the optimal value is one, then DMU_o is called Farrell-efficient (weakly efficient); otherwise, DMU_o is called a purely technically inefficiency unit (Cooper et al., 2007).

3. New Non-Radial Multiplicative Model

Suppose that for all $i = 1,2,\ldots,m$ and $r = 1,2,\ldots,s$, unit $(x_{io}^{1-\alpha_i}, y_{ro}^{1+\beta_r})$ dominates $DMU_O = (X_O, Y_O)$; therefore, $1 \leqslant x_{io}^{1-\alpha_i} \leqslant x_{io}$, $1 \leqslant y_{ro} \leqslant y_{ro}^{1+\beta_r}$, such that $0 \leqslant \alpha_i \leqslant 1$ and $\beta_r \geqslant 0$. To find the optimal value of α_i and β_r for all i and r, such that $(x_{io}^{1-\alpha_i}, y_{ro}^{1+\beta_r}) \in PPS_{(GVRS)}$, we introduce the following multi-objective nonlinear programming:

$$
\begin{aligned}
&Min \quad 1-\alpha_i, \quad for\, i=1,2,...,m \\
&Max \quad 1+\beta_r, \quad for\, r=1,2,...,s \\
&S.to: \quad \left(x_{io}^{1-\alpha_i}, y_{ro}^{1+\beta_r}\right) \in PPS_{(GVRS)}, \quad i=1,2,...,m,\ r=1,2,...,s; \\
&\qquad 0 \leqslant \alpha_i \leqslant 1, \quad i=1,2,...,m; \\
&\qquad \beta_r \geqslant 1, \quad r=1,2,...,s.
\end{aligned} \tag{2}
$$

Model (2) calculates the optimal values of α_i $(for\ i = 1, 2, \ldots, m)$ and β_r $(for\ r = 1, 2, \ldots, s)$ so that the new unit $(x_{io}^{1-\alpha_i}, y_{ro}^{1+\beta_r})$ lies in $PPS_{(GVRS)}$ and dominates $DMU_O = (X_O, Y_O)$. To solve (2), by using the $PPS_{(GVRS)}$ structure, we convert it to the following single-objective model:

$$
\begin{aligned}
&Min \quad \rho_e = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\alpha_i}{1+\frac{1}{s}\sum_{r=1}^{s}\beta_r} \\
&S.to: \quad \prod_{j=1, j\neq o}^{n} x_{ij}^{\lambda_j} . x_{io}^{\lambda_o} \leqslant x_{io}^{(1-\alpha_i)}, \quad i=1,2,...,m; \\
&\qquad \prod_{j=1, j\neq o}^{n} y_{rj}^{\lambda_j} . y_{ro}^{\lambda_o} \geqslant y_{ro}^{(1+\beta_r)}, \quad r=1,2,...,s; \\
&\qquad \sum_{j=1, j\neq o}^{n} \lambda_j + \lambda_o = 1; \\
&\qquad 0 \leqslant \alpha_i \leqslant 1, \quad i=1,2,...,m; \\
&\qquad \beta_r \geqslant 0, \quad r=1,2,...,s; \\
&\qquad \lambda_j \geqslant 0, \lambda_o \geqslant 0, \quad j=1,2,...,n,\ j \neq o.
\end{aligned} \tag{3}
$$

Model (3) evaluates unit $DMU_o = (X_o, Y_o)$ as non-radial and input and output-oriented and also determines whether DMU_o is a Pareto-efficient or inefficient unit. Model (3) is feasible, as for all $i = 1, 2, \ldots, m$, $r = 1,2, \ldots, s$, and $j = 1, 2, \ldots, n$, the solution $(\alpha_i = 0, \beta_r = 0, \lambda_j = 0, \lambda_o = 1)$ is its feasible solution. Additionally, since for all i and r, $0 \leqslant \alpha_i \leqslant 1$ and $\beta_r \geqslant 0$, the objective function value of (3) satisfies $0 \leqslant \rho \leqslant 1$.

Remark 3.1

For all i and r, if α_i^* and β_r^* are the optimal solutions of (3), then the efficiency score of *i*-th input and *r*-th output of DMU_o are $1 - \alpha_i^*$ and $1 + \beta_r^*$, respectively (input and output efficiency refers to the amount of input consumption and output production that a unit can be converted into a Pareto-optimal unit (Cooper et al., 2007)); hence, for DMU_o, the required values of *i*-th input and *r*-th output to achieve a Pareto-efficient are $x_{io}^{1-\alpha_i^*}$ and $y_{ro}^{1+\beta_r^*}$, respectively. Also, the optimal value of (3) is the efficiency score of DMU_o, which is $\rho_e^* = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\alpha_i^*}{1+\frac{1}{s}\sum_{r=1}^{s}\beta_r^*}$. Moreover, we will show in the next theorem that, if α_i^* and β_r^*, for all i and r, are zero $(i.e.\, \rho_e^* = 1)$, then DMU_o is Pareto-efficient.

Theorem 3.2

Suppose for $i = 1, 2, \ldots, m$, $r = 1,2, \ldots, s$, and $j = 1, 2, \ldots, n$, $(\alpha_i^*, \beta_r^*, \lambda_j^*)$ is the optimal solution of (3). Then DMU_o is Pareto-efficient if and only if $\alpha_i^* = 0$ and $\beta_r^* = 0$ for all i and r.

Proof. By contradiction, suppose DMU_o is Pareto-efficient, and there exists t such that $\alpha_t^* > 0$ and for all $i \neq t$, and r, $\alpha_i^* = 0$, and $\beta_r^* = 0$. Therefore, according to the constraints of (3), we have:

$$\prod_{j=1, j\neq o}^{n} X_j^{\lambda_j^*} . X_o^{\lambda_o^*} < X_o, \quad \prod_{j=1, j\neq o}^{n} Y_j^{\lambda_j^*} . Y_o^{\lambda_o^*} \geqslant Y_o. \tag{4}$$

Therefore, $(-\prod_{j=1,j\neq o}^{n} X_j^{\lambda_j^*} . X_o^{\lambda_o^*}, \prod_{j=1,j\neq o}^{n} Y_j^{\lambda_j^*} . Y_o^{\lambda_o^*})$ dominates $(-X_o, Y_o)$ which contradicts the Pareto-efficiency property of DMU_o. On the other hand, suppose for all $i = 1, 2, \ldots, m, r = 1, 2, \ldots, s, \alpha_i^* = 0$ and $\beta_r^* = 0$; we prove that DMU_o is Pareto-efficient. Again, by contradiction, suppose that DMU_o is not Pareto-efficient; therefore, there exists

$$\hat{\lambda} \geqslant 0 \text{ and } \sum_{j=1, j\neq o}^{n} \hat{\lambda}_j + \hat{\lambda}_o = 1$$

such that $(-\prod_{j=1,j\neq o}^{n} X_j^{\hat{\lambda}_j} . X_o^{\hat{\lambda}_o}, \prod_{j=1,j\neq o}^{n} Y_j^{\hat{\lambda}} . Y_o^{\hat{\lambda}_o})$ dominates $(-X_o, Y_o)$. Thus, we have

$$\prod_{j=1, j\neq o}^{n} X_j^{\hat{\lambda}_j} . X_o^{\hat{\lambda}_o} \leqslant X_o, \quad \prod_{j=1, j\neq o}^{n} Y_j^{\hat{\lambda}} . Y_o^{\hat{\lambda}_o} \geqslant Y_o$$

and at least one inequality is strict. Let $1 \leqslant t \leqslant m$ be such that

$$\prod_{j=1, j\neq o}^{n} x_{tj}^{\hat{\lambda}_j} . X_o^{\hat{\lambda}_o} < x_{to}$$

and for all $i \neq t$ and r,

$$\prod_{j=1, j\neq o}^{n} x_{ij}^{\hat{\lambda}_j} . x_{io}^{\hat{\lambda}_o} \leqslant x_{io} \text{ and } \prod_{j=1, j\neq o}^{n} y_{rj}^{\hat{\lambda}} . y_{ro}^{\hat{\lambda}_o} \geqslant y_{ro}$$

Therefore, $(\hat{\lambda}, \hat{\alpha}_t > 0, \hat{\alpha}_i = 0 (i \neq t), \hat{\beta}_r = 0)$ is a feasible solution for (3) and the value of the objective function at this solution is less than the optimal value of (3); this contradicts our assumption that $(\alpha^* = 0, \beta^* = 0)$ is an optimal solution of (3).

Remark 3.3

When evaluating DMU_o by model (3), for all i and r, if (α_i^*, β_r^*) is the optimal solution, then the new unit $(x_{io}^{1-\alpha_i^*}, y_{ro}^{1+\beta_r^*})$ is the improvement of DMU_o, and it is Pareto-efficient.

Proposition 3.4

In evaluating DMU_o, suppose that θ^* and ρ_e^* are the optimal values of (1) and (3), respectively; then $\rho_e^* \leqslant \theta^*$.

Proof. According to the feasible set of the two models, if (λ^*, θ^*) is the optimal solution of (1), then for all i and r, $(\lambda^*, \alpha_i = 1 - \theta^*, \beta_r = 0)$ is a feasible solution of (3). Hence, we have:

$$\rho_e^* = Min \frac{1-\frac{1}{m}\sum_{i=1}^{m}\alpha_i}{1+\frac{1}{s}\sum_{r=1}^{s}\beta_r} \leqslant \frac{1-\frac{1}{m}\sum_{i=1}^{m}\left(1-\theta^*\right)}{1} = \theta^* \tag{5}$$

Proposition 3.4 reflects that ρ_e^* accounts for all inefficiencies, whereas θ^* accounts only for purely technically inefficiencies. Furthermore, in evaluating DMU_o by (3), if $\rho_e^* = 1$, then $\theta^* = 1$ and DMU_o based on model (1) is also a Pareto-efficient unit.

4. A New Approach to Merge

This study proposes a new non-radial merger model derived from Model (3), enabling managers to control and manage the efficiency of all inputs and outputs of the newly obtained unit. To emphasize the importance of this model, a brief review of existing DEA-based merger models is first provided.

4-1. Brief Review of Some DEA Merger Models

Typically, when considering a merger, it is assumed that the input or output indicators of the merged units are summed together. However, such a merger is often not beneficial, since the resulting unit is not necessarily efficient; in fact, it is sometimes observed that merging two efficient units does not yield an efficient one (Tone, 1996). In practice, a merger is successful only if the newly formed unit is efficient or meets the expected efficiency target.

Gattoufi et al. (2014) introduced model (6) for merging two units within $PPS_{(VRS)}$ using Inv DEA. The new unit formed by combining units *P* and *K* is referred to as *M*, whose output equals the combined outputs of *P* and *K*:

$$\begin{aligned}
Min \quad & \sum_{i=1}^{m}\left(a_{iK}+a_{iP}\right) \\
S.to: \quad & \sum_{j\in F}\lambda_j x_{ij} + \lambda_M\left(x_{iK}+x_{iP}\right) \leqslant \bar{\theta}\left(a_{iK}+a_{iP}\right), \qquad i=1,2,...,m; \\
& \sum_{j\in F}\lambda_j y_{rj} + \lambda_M\left(y_{rK}+y_{rP}\right) \geqslant \left(y_{rk}+y_{rP}\right), \qquad r=1,2,...,s; \\
& \sum_{j\in F}\lambda_j + \lambda_M = 1; \\
& \lambda_M \geqslant 0, \lambda_j \geqslant 0; \qquad \forall j \in F; \\
& 0 \leqslant a_{iK} \leqslant x_{iK}, 0 \leqslant a_{iP} \leqslant x_{iP}, \qquad i=1,2,...,m,
\end{aligned} \tag{6}$$

where a_{iK} and a_{iP} indicate the *i*-th input values of units *P* and *K* post-merger, *F* represents the set of DMUs after the merger excluding the merged units, and $\bar{\theta}$ is the predetermined efficiency score of the unit resulting from the merger.

The formulation in (6) is nonlinear; Gattoufi et al. (2014) converted it into a linear one by considering $\lambda_M = 0$ and $\lambda_M = 1$. However, this may render problem (6) infeasible. For instance, consider the five units presented in Table 1 with a single input and a single output.

Table 1. Data of Five Sample Units with One Input and One Output

DMU	A	B	C	D	E
I	1.0	2.0	3.0	2.0	4.0
O	2.0	3.0	3.0	2.0	4.0

For $\bar{\theta} \leqslant 1$ and $\lambda_M = 0$, model (6) has no solution when merging two units *C* and *D*. Similarly, assuming $\lambda_M = 1$ and $\bar{\theta} \leqslant 1$, model (6) remains infeasible. The production possibility set in model (6) has convex arithmetic and variable returns to scale technology. Also, model (6) is radial and input-oriented; in other words, after the merger, only the inputs of the units are optimized.

Amin & Ibn Boamah (2020) addressed the infeasibility of model (6) by modifying its feasible set. They proposed a model in which the merged unit is determined at the minimum cost, taking into account the input prices of the merged units. The model presented by Amin & Ibn Boamah (2020) is as follows:

$$
\begin{aligned}
&Min \quad \sum_{i=1}^{m} c_i \left(a_{iK} + a_{iP}\right) \\
&S.to: \quad \sum_{j\in F} \lambda_j x_{ij} + \lambda_M \left(x_{iK} + x_{iP}\right) \leqslant \left(a_{iK} + a_{iP}\right), \qquad i = 1,2,...,m; \\
&\qquad \sum_{j\in F} \lambda_j y_{rj} + \lambda_M \left(y_{rK} + y_{rP}\right) \geqslant \left(y_{rk} + y_{rP}\right), \qquad r = 1,2,...,s; \\
&\qquad \sum_{i=1}^{m} c_i \left(a_{iK} + a_{iP}\right) = \frac{1}{\bar{C}} \sum_{i=1}^{m} c_i \left(\bar{a}_{iK} + \bar{a}_{iP}\right); \\
&\qquad \sum_{j\in F} \lambda_j + \lambda_M = 1; \\
&\qquad \lambda_M \geqslant 0,\ \lambda_j \geqslant 0; \qquad \forall j \in F; \\
&\qquad 0 \leqslant a_{iK} \leqslant x_{iK}, 0 \leqslant a_{iP} \leqslant x_{iP}, \qquad \forall i,
\end{aligned}
\tag{7}
$$

where $\bar{C}$ is the predetermined cost efficiency of the new unit, and for each $i = 1, 2, \ldots, m$, c_i denotes the price of *i*-th input, and $\bar{a}_{iK}$, $\bar{a}_{iP}$ are the optimal input values obtained from the model proposed by Cooper et al. (2007):

$$
\begin{aligned}
&Min \quad \sum_{i=1}^{m} c_i \left(a_{iK} + a_{iP}\right) \\
&S.to: \quad \sum_{j\in F} \lambda_j x_{ij} + \lambda_M \left(x_{iK} + x_{iP}\right) \leqslant \left(a_{iK} + a_{iP}\right), \qquad i = 1,2,...,m; \\
&\qquad \sum_{j\in F} \lambda_j y_{rj} + \lambda_M \left(y_{rK} + y_{rP}\right) \geqslant \left(y_{rk} + y_{rP}\right), \qquad r = 1,2,...,s; \\
&\qquad \sum_{j\in F} \lambda_j + \lambda_M = 1; \\
&\qquad \lambda_M \geqslant 0,\ \lambda_j \geqslant 0; \qquad \forall j \in F; \\
&\qquad 0 \leqslant a_{iK} \leqslant x_{iK},\ 0 \leqslant a_{iP} \leqslant x_{iP}, \qquad i = 1,2,...,m.
\end{aligned}
\tag{8}
$$

Problem (7) can also become infeasible for numerous choices of $\bar{C}$. For instance, consider the five units with a single input and a single output, as presented in Table 2.

Table 2. Data of Five Sample Units with One Input and One Output

DMU	A	B	C	D	E
I	1.0	2.0	3.0	2.0	4.0
O	2.0	3.0	3.0	1.0	4.0

If DMU_C and DMU_D, in Table 2, are merged by model (7), for input price $c_1 = 1$ and $\bar{C} < 0.80$, model (7) is infeasible. Also, if DMU_B and DMU_C are merged with the input price $c_1 = 1$ and the cost efficiency $\bar{C} < 1$, model (7) becomes infeasible. Moreover, for $c_1 = 1$ and $\bar{C} = 1$, if units *A*

and *C* are merged using models (7), after merging, the input of DMU_A becomes zero; in other words, unit *A* has no input contribution in this merger, and practically, the merger will be unsuccessful. Similar to the model of Gattoufi et al. (2014), the model of Amin & Ibn Boamah (2020) is radial and input-oriented, and its production possibility set has convex arithmetic and variable returns to scale technology.

Soltanifar et al. (2024) merged units *p* and *k* using the common set of weights (CSW) based on the following goal programming.

$$
\begin{aligned}
&Min \sum_{j\in\{1,2,\ldots,n\}-\{p,k\}\cup\{G\}} d_j \\
&S.to: \sum_{i=1}^{s} u_r x_{rj} - E_j^* \sum_{i=1}^{m} v_i x_{ij} + d_j = 0, \quad j \in \{1,2,\ldots,n\} - \{p,k\}; \\
&\sum_{i=1}^{s} u_r \left(\beta_{rp} + \beta_{rk}\right) - E_G^* \sum_{i=1}^{m} v_i \left(\alpha_{ip} + \alpha_{ik}\right) + d_G = 0; \\
&0 \le \alpha_{ip} + \alpha_{ik} \le x_{ip} + x_{ik}, \quad i \in \{1,2,\ldots,m\}; \\
&0 \le y_{rp} + y_{rk} \le \beta_{rp} + \beta_{rk}, \quad r \in \{1,2,\ldots,s\}; \\
&d_j \ge 0, \quad j \in \{1,2,\ldots,n\} - \{p,k\} \cup \{G\}; \\
&v_i \ge 0, \quad i \in \{1,2,\ldots,m\}; \\
&u_r \ge 0, \quad r \in \{1,2,\ldots,s\},
\end{aligned}
\tag{9}
$$

where for $i = 1,2,\ldots,m$, α_{ip} and α_{ik} denote the inputs of units *p* and *k* after merger, respectively. Also, for $r = 1,2,\ldots,s$, β_{rp} and β_{rk} denote the outputs of units *p* and *k* after merger, respectively. For $j \in \{1,2,\ldots,n\} - \{p,k\}$, E_j^* is CCR efficiency of DMU_j. G is the obtained unit from the merger of units *p* and *k*, and E_G^* denotes its predetermined efficiency. The CSW model of Soltanifar et al. (2024) is input and output-oriented, and similar to the models of Gattoufi et al. (2014) and Amin & Ibn Boamah (2020), its production possibility set is based on convex arithmetic technology; hence, their efficient frontiers are union segments.

We aim to develop a merger model that performs the merger process with greater precision compared to the existing approaches. Since the efficient frontier in the production possibility set with a convex geometric and variable returns to scale technology is represented by a curve, it is expected that the obtained efficient unit from the merger will be located more accurately on this frontier (Kao, 2023). In the previously proposed models, such as Gattoufi et al. (2014), Amin & Ibn Boamah (2020), and Soltanifar et al. (2024), managers radially reach their expected unit in the merger. In contrast, our proposed model is designed to operate as non-radial, meaning that managers can monitor the efficiency of all inputs and outputs simultaneously. Furthermore, unlike the model of Gattoufi et al. (2014) and Amin & Ibn Boamah (2020) and similar to the model of Soltanifar et al. (2024), the proposed framework ensures that both inputs and outputs of the merged units are optimized simultaneously. Another limitation of the prior models is that some units may have no share of inputs or outputs in the merger. To address this, our model guarantees that all units contribute to the merger, and no input or output will take a zero value after the merger. Moreover, the proposed model is always feasible.

4-2. Introducing the New Multiplicative Merger Model

Suppose that the set E contains units that we incline to merge. First, we consider unit M as $DMU_M = (\sum_{k\in E} x_{ik}, \sum_{k\in E} y_{rk})$ and add it to $PPS_{(GVRS)}$; hence $PPS_{(GVRS)}$ is extend to

$$
PPS_{(GVRS)_{new}} = \{(X,Y) \mid \prod_{j=1}^{n} X_j^{\lambda_j} . X_M^{\lambda_M} \leqslant X, \prod_{j=1}^{n} Y_j^{\lambda_j} . Y_M^{\lambda_M} \geqslant Y, 1\lambda = 1, \lambda \geqslant 0\}.
$$

Now, we evaluate DMU_M in $PPS_{(GVRS)_{new}}$, by model (3). Accordingly, suppose that for all $i = 1,2,\ldots,m$ and $r = 1,2,\ldots,s$, $DMU_N = ((x_{iM}^{1-\alpha_i})^{\bar{\theta}_i}, (y_{rM}^{1+\beta_r})^{\bar{\varphi}_r})$ dominates DMU_M. Therefore,

$$1\leqslant(x_{iM}^{1-\alpha_i})^{\bar{\theta}_i}\leqslant x_{iM},\ 1\leqslant y_{rM}\leqslant(y_{rM}^{1+\beta_r})^{\bar{\varphi}_r}$$

such that $0 \leq \bar{\theta}_i \leq 1$, $\bar{\varphi}_r \geq 1$ are the defined efficiency scores for *i*-th input and *r*-th output of the new unit obtained from the merger, respectively, and $0 \leqslant \alpha_i \leqslant 1$, $\beta_r \geqslant 0$ are variables. To find the optimal values of α_i and β_r such that $DMU_N = ((x_{iM}^{1-\alpha_i})^{\bar{\theta}_i}, (y_{rM}^{1+\beta_r})^{\bar{\varphi}_r})$ remains in $PPS_{(GVRS)_{new}}$ and dominate DMU_M, model (3) is converted to the following model (10):

$$Min \quad \rho_m = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\alpha_i}{1+\frac{1}{s}\sum_{r=1}^{s}\beta_r}$$

$$S.to: \quad \prod_{j=1}^{n} x_{ij}^{\lambda_j}.x_{iM}^{\lambda_M} \leqslant x_{iM}^{\bar{\theta}_i(1-\alpha_i)}, \qquad i=1,2,...,m;$$

$$\prod_{j=1}^{n} y_{rj}^{\lambda_j}.y_{rM}^{\lambda_M} \geqslant y_{rM}^{\bar{\varphi}_r(1+\beta_r)}, \qquad r=1,2,...,s; \tag{10}$$

$$\sum_{j=1}^{n}\lambda_j + \lambda_M = 1;$$

$$0\leqslant\alpha_i\leqslant 1, \qquad i=1,2,...,m;$$

$$\beta_r\geqslant 0, \qquad r=1,2,...,s;$$

$$\lambda_j\geqslant 0, \lambda_M\geqslant 0 \qquad j=1,2,...,n.$$

Suppose that α_i^* and β_r^* are the optimal solutions of (10) for all $i = 1,2,\dots,m$ and $r = 1,2,\dots,s$, considering the predetermined efficiency scores $0 \leq \bar{\theta}_i \leq 1$ and $\bar{\varphi}_r \geq 1$ for *i*-th input and *r*-th output, respectively. We will show that $DMU_G = (x_{iM}^{1-\alpha_i^*}, y_{rM}^{1+\beta_r^*})$ is the obtained unit from merging units of E and its efficiency score is $\bar{\theta} = \frac{s\sum_{i=1}^{m}\bar{\theta}_i}{m\sum_{r=1}^{s}\bar{\varphi}_r}$.

Proposition 4.1

Suppose for all i and r, α_i^* and β_r^* are the optimal solutions of (10) when $\bar{\theta}_i = 1$ and $\bar{\varphi}_r = 1$. Then, $1 - \alpha_i^* \leqslant \bar{\theta}_i \leqslant 1$ and $1 + \beta_r^* \geqslant \bar{\varphi}_r \geqslant 1$ for all i and r, and model (10) is always feasible.

Proof. According to the Remark 3.3, $(x_{iM}^{(1-\alpha_i^*)}, y_{rM}^{(1+\beta_r^*)})$ is Pareto-efficient; therefore, for all i and r, we have

$$x_{iM}^{\left(1-\alpha_i^*\right)}\leqslant x_{iM}^{\bar{\theta}_i(1-\alpha_i)} \text{ and } y_{rM}^{\left(1+\beta_r^*\right)}\geqslant y_{rM}^{\bar{\varphi}(1+\beta_r)}$$

Since $X_M \geqslant 1$ and $Y_M \geqslant 1$,for all i and r we have $1 - \alpha_i^* \leqslant \bar{\theta}_i(1 - \alpha_i)$ and $1 + \beta_r^* \geqslant \bar{\varphi}_r(1 + \beta_r)$. Now, because $0 \leqslant 1 - \alpha_i \leqslant 1$ and $1 + \beta_r \geqslant 1$, for all $i = 1, 2, \dots, m$ and $r = 1,2, \dots, s$, we have

$$1 - \alpha_i^* \leqslant \bar{\theta}_i(1 - \alpha_i) \leqslant \bar{\theta}_i \leqslant 1 \text{ and } 1 + \beta_r^* \geqslant \bar{\varphi}_r(1 + \beta_r) \geqslant \bar{\varphi}_r \geqslant 1$$

On the other hand, in model (10), by considering $1 - \alpha_i^* \leqslant \bar{\theta}_i \leqslant 1$ and $1 + \beta_r^* \geqslant \bar{\varphi}_r \geqslant 1$, for all i and r, we have

$$x_{iM}^{\bar{\theta}_i(1-\alpha_i)}\leqslant x_{iM}^{(1-\alpha_i)} \text{ and } y_{rM}^{\bar{\varphi}_r(1+\beta_r)}\geqslant y_{rM}^{(1+\beta_r)}$$

Therefore, for all $j = 1,2,\dots,n$, $r = 1,2,\dots,s$, and $i = 1, 2, \dots, m$, $(\alpha_i = 0, \beta_r = 0, \lambda_j = 0, \lambda_M = 1)$ is a feasible solution for (10).

4-3. Advantages of Merging Based on Model (10)

Suppose for all $i = 1, 2, \dots, m$, $r = 1,2,\dots,s$, and $j = 1,2,\dots,n$, λ_j^*, λ_M^*, α_i^* and β_r^* are the optimal solutions of (10); then, the following six results are obtained after merging by model (10):

1. Since in the optimal condition, the first and second sets of constraints of (10) are equalities, $DMU_N = (x_{iM}^{(1-\alpha_i^*)\bar{\theta}_i}, y_{rM}^{(1+\beta_r^*)\bar{\varphi}_r})$ lies on the efficient frontier and is Pareto-efficient.

2. In model (10), for all $i = 1,2,\dots,m$ and $r = 1,2,\dots,s$, consider $\bar{\theta}_i = 1$ and $\bar{\varphi}_r = 1$; if we have $\alpha_i^* = 0$, $\beta_r^* = 0$, then, according to Theorem 3.2, $DMU_N = DMU_M$ is Pareto-efficient. Otherwise, $DMU_G = (x_{iM}^{1-\alpha_i^*}, y_{rM}^{1+\beta_r^*})(for\ i = 1,2,\dots,m, r = 1,2,\dots,s)$ is the improved version of DMU_M, and it is the newly obtained unit from the merger of units in E, which is Pareto-efficient.

3. For a vector $(\bar{\theta}, \bar{\varphi}) \neq 1$ in (10), a new unit $DMU_G = (x_{iM}^{1-\alpha_i^*}, y_{rM}^{1+\beta_r^*})(for\ i = 1,2,\dots,m, r = 1,2,\dots,s)$, with the efficiency score $\bar{\theta} = \frac{s\sum_{i=1}^{m}\bar{\theta}_i}{m\sum_{r=1}^{s}\bar{\varphi}_r}$ is obtained.

To show advantage 3, we employ model (3) to evaluate $DMU_G = (x_{iM}^{1-\alpha_i^*}, y_{rM}^{1+\beta_r^*})(i = 1,2,\dots,m, r = 1,2,\dots,s)$ in $PPS_{(GVRS)_{new}}$; therefore, we have:

$$Min \quad \rho_e = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\delta_i}{1+\frac{1}{s}\sum_{r=1}^{s}\gamma_r}$$

$$\begin{aligned} S.to:\ & \prod_{j=1}^{n} x_{ij}^{\lambda_j} . x_{iM}^{\lambda_M} \leqslant (x_{iM}^{1-\alpha_i^*})^{1-\delta_i}, \qquad i = 1,2,...,m; \\ & \prod_{j=1}^{n} y_{rj}^{\lambda_j} . y_{rM}^{\lambda_M} \geqslant (y_{rM}^{1+\beta_r^*})^{1+\gamma_r}, \qquad r = 1,2,...,s; \\ & \sum_{j=1}^{n}\lambda_j + \lambda_M = 1; \\ & 0 \leqslant \delta_i \leqslant 1, \qquad i = 1,2,...,m; \\ & \gamma_r \geqslant 0, \qquad r = 1,2,...,s; \\ & \lambda_j \geqslant 0,\ \lambda_M \geqslant 0 \qquad j = 1,2,...,n. \end{aligned} \tag{11}$$

where, for all i and r, we have used δ_i and γ_r instead of α_i and β_r, respectively, in model (3).

By contradiction, suppose that, for all i, r, and j, the amounts of δ_i^*, γ_r^* and $\bar{\lambda}^*$ are the optimal solution of model (11); moreover, ρ_e^* is the efficiency score of $DMU_G = (x_{iM}^{1-\alpha_i^*},\ y_{rM}^{1+\beta_r^*})$ such that

$$\rho_e^* = \frac{1-\frac{1}{m}\sum_{i=1}^{m}\delta_i^*}{1-\frac{1}{s}\sum_{r=1}^{s}\gamma_r^*} < \bar{\theta} = \frac{s\sum_{i=1}^{m}\bar{\theta}_i}{m\sum_{r=1}^{s}\bar{\varphi}_r}$$

Let there exist t such that $1-\delta_t^* < \bar{\theta}_t$, and for all $i \neq t$ and $r = 1,2,\dots,s$, $1-\delta_i^* = \bar{\theta}_i$ and $1+\gamma_r^* = \bar{\varphi}_r$. Now, since, in the optimal condition, the first and second sets of constraints in (11) are equal, for $i = t$,

$$\prod_{j=1}^{n} x_{tj}^{\bar{\lambda}_j^*} . x_{tM}^{\bar{\lambda}_M^*} = x_{tM}^{(1-\alpha_t^*)(1-\delta_i^*)} < x_{tM}^{(1-\alpha_t^*)(\bar{\theta}_t)}$$

and for all $i \neq t$ and $r = 1,2,\dots,s$, we have

$\prod_{j=1}^{n} x_{ij}^{\bar{\lambda}_j^*} . x_{iM}^{\bar{\lambda}_M^*} = x_{iM}^{(1-\alpha_i^*)(1-\delta_i^*)} = x_{iM}^{(1-\alpha_i^*)(\bar{\theta}_i)}$ and also $\prod_{j=1}^{n} y_{rj}^{\bar{\lambda}_j^*} . y_{rM}^{\bar{\lambda}_M^*} = y_{rM}^{(1+\beta_r^*)(1+\gamma_r^*)} = y_{rM}^{\bar{\varphi}_r(1+\beta_r^*)}$

Therefore, $(\prod_{j=1}^{n} x_{ij}^{\bar{\lambda}_j^*} . x_{iM}^{\bar{\lambda}_M^*}, \prod_{j=1}^{n} y_{rj}^{\bar{\lambda}_j^*} . y_{rM}^{\bar{\lambda}_M^*})$ dominates the Pareto-efficient unit $(x_{iM}^{(1-\alpha_i^*)\bar{\theta}_i}, y_{rM}^{(1+\beta_r^*)\bar{\varphi}_r})$, which, according to advantage 1, contradicts its Pareto-efficiency.

4. In DMU_G, i-th input and r-th output are denoted by $x_{iM}^{-\alpha_i^*} . x_{iM}$ and $y_{rM}^{\beta^*} . y_{rM}$, respectively. Since $x_{iM} = \sum_{k\in E} x_{ik}$ and $y_{rM} = \sum_{k\in E} y_{rk}$, the contribution of i-th input and r-th output of each DMU_k in E

(for $k \in E$) is equal to $(\frac{1}{x_{iM}})^{\alpha_i^*} x_{ik}$ and $y_{rM}^{\beta_r^*} y_{rk}$, respectively; this indicates that the participation ratio of each input is equal in all units, and the participation ratio of each output is equal in all units as well.

5. Since, for all i and r, $x_{iM} \geqslant 1$ and $y_{rM} \geqslant 1$, we have $(\frac{1}{x_{iM}})^{\alpha_i^*} \neq 0$ and $y_{rM}^{\beta_r^*} \neq 0$, the contribution of any of the indices (input and output) of the merged units would not be zero, and each merged unit would participate in the merger.

6. Since for all $i = 1,2,\ldots,m$ and $r = 1,2,\ldots,s$, $DMU_N = (x_{iM}^{(1-\alpha_i^*)\bar{\theta}_i}, y_{rM}^{(1+\beta_r^*)\bar{\varphi}_r})$ is Pareto-efficient, as $\bar{\theta}_i$ decreases, $(1-\alpha_i^*)$ increases, and consequently, α_i^* decreases. Hence, based on $(\frac{1}{x_{iM}})^{\alpha_i^*} x_{ik}$, the share value of the *i*-th input of unit k increases. Also, as $\bar{\varphi}_r$ increases, $(1+\beta_r^*)$ decreases, and consequently, β_r^* decreases. Hence, based on $y_{rM}^{\beta_r^*} y_{rk}$, the share value of the *r*-th output of unit k decreases.

4-4. Solving the New Evaluation and Merger Models

To solve the nonlinear models (3) and (10), we apply the natural logarithm and use Charnes et al.'s (1962) method by choosing $t \equiv \frac{1}{1+\frac{1}{s}\sum_{r=1}^{s}\beta_r} > 0$, $a \equiv t\alpha$, $b \equiv t\beta$, and $\Lambda \equiv t\lambda$; then, one can convert them to linear models (12) and (13), respectively, as follow:

$$
\begin{aligned}
&Min \quad \rho_e = t - \frac{1}{m}\sum_{i=1}^{m} a_i \\
&S.to: \quad \sum_{j=1}^{n} \Lambda_j \bar{x}_{ij} \leqslant (t - a_i)\bar{x}_{io}, \qquad \forall i; \\
&\qquad \sum_{j=1}^{n} \Lambda_j \bar{y}_{rj} \geqslant (t + b_r)\bar{y}_{ro}, \qquad \forall r; \\
&\qquad \sum_{j=1}^{n} \Lambda_j = t; \\
&\qquad t + \frac{1}{s}\sum_{r=1}^{s} b_r = 1; \\
&\qquad a_i \geqslant 0, \qquad \forall i; \\
&\qquad t - a_i \geqslant 0, \qquad \forall i; \\
&\qquad b_r \geqslant 0, \qquad \forall r; \\
&\qquad \Lambda_j \geqslant 0, \qquad \forall j;
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&Min \quad \rho_m = t - \frac{1}{m}\sum_{i=1}^{m} a_i \\
&S.to: \quad \sum_{j=1}^{n} \Lambda_j \bar{x}_{ij} + \Lambda_M \bar{x}_{iM} \leqslant \bar{\theta}_i (t - a_i)\bar{x}_{iM}, \qquad \forall i; \\
&\qquad \sum_{j=1}^{n} \Lambda_j \bar{y}_{rj} + \Lambda_M \bar{y}_{rM} \geqslant \bar{\varphi}_r (t + b_r)\bar{y}_{rM}, \qquad \forall r; \\
&\qquad \sum_{j=1}^{n} \Lambda_j + \Lambda_M = t; \\
&\qquad t + \frac{1}{s}\sum_{r=1}^{s} b_r = 1; \\
&\qquad a_i \geqslant 0, \qquad \forall i; \\
&\qquad t - a_i \geqslant 0, \qquad \forall i; \\
&\qquad b_r \geqslant 0, \qquad \forall r; \\
&\qquad \Lambda_j \geqslant 0, \Lambda_M \geqslant 0, \qquad \forall j;
\end{aligned}
\tag{13}
$$

where, for all i, r and j, $\bar{x}_{ij} = lnx_{ij} \geqslant 0$, $\bar{x}_{iM} = lnx_{iM} \geqslant 0$, $\bar{y}_{rj} = lny_{rj} \geqslant 0$, and $\bar{y}_{rM} = lny_{rM} \geqslant 0$.

Now, we solve linear models (12) and (13) instead of solving nonlinear models (3) and (10). Models (12) and (13) evaluate and merge units within PPS with convex geometric and variable returns to scale technology, respectively. If $(\Lambda^*, a^*, b^*, t^* > 0)$ be the optimal solution of (12), then $(\lambda^* = \frac{\Lambda^*}{t^*}, \alpha^* = \frac{a^*}{t^*}, \beta^* = \frac{b^*}{t^*})$ is the optimal solution of model (3). And, if $(\Lambda^*, a^*, b^*, t^* > 0)$ be the optimal solution of (13), then $(\lambda^* = \frac{\Lambda^*}{t^*}, \alpha^* = \frac{a^*}{t^*}, \beta^* = \frac{b^*}{t^*})$ is the optimal solution of model (10).

Remark 4.2

Both sides of the first inequalities of (12) and (13) are positive; therefore, for all $r = 1,2,\ldots,s$, constraints $(t - a_i) \geqslant 0$ are always valid, and hence, one can remove them from the two models.

5. Comparison of Models

In this section, we compare the newly proposed evaluation and merger models with other existing models in these fields.

5-1. Comparison of Models (1), (3) and BCC

1. The efficient frontier of PPS with convex geometry technology is a non-concave curve (Cooper et al., 2007) and multiplicative models provide a proper and better measure of the efficiency of DMUs compared to non-multiplicative models (Kao, 2023). Accordingly, model (3) is more applicable for evaluating units in PPS with convex geometry technology than non-multiplicative models such as the BCC model. Additionally, as shown in Section 3, model (3) evaluates units as non-radial and identifies Pareto-efficient and all inefficient units. In contrast, in model (1), the evaluation is done radially and identifies weakly efficient and purely technically inefficient units.
2. According to Remark 3.1, in evaluating DMU_o by model (3), the efficiency score of inputs and outputs of DMU_o is also obtained, and the improved unit of DMU_o will be a Pareto-efficient unit. In contrast, in model (1), the efficiency score of inputs and outputs of DMU_o could not be obtained, and the improved unit of DMU_o will be weakly efficient.

The above points are summarized in the Table 3.

Table 3. Comparisons of Models (1), (3) and BCC

Model	PPS technology	Efficient frontier	Model type	Efficiency score of inputs and outputs	Efficiency status	Identifying inefficient units
Model BCC	convex arithmetic	union of segments	radial	incalculable	weakly efficient	technically inefficient units
Model (1)	convex geometry	curve	radial	incalculable	weakly efficient	technically inefficient units
Model (3)	convex geometry	curve	non-radial	computable	Pareto-efficient	all inefficient units

As shown in Proposition 3.4, let θ^* and ρ_e^* be the efficiency scores of DMU_o by models of (1) and (3), respectively; then $\rho_e^* \leqslant \theta^*$.

For instance, six units with one input and one output are presented in Table 4. The efficiency score of each unit is calculated using models (1) and (3) and added to the table.

Table 4. Data of Six Sample Units with One Input and One Output

DMU	A	B	C	D	E	F
I	2.0	3.0	4.0	5.0	6.0	7.0
O	3.0	4.0	2.0	2.0	5.0	5.0
θ^*	1.0	1.0	0.5	0.43	1.0	0.92
ρ_e^*	1.0	1.0	0.31	0.27	1.0	0.92

As presented in Table 4, the obtained efficiency from model (3) is more accurate than that of model (1). Using model (3) and Remark 3.1, the input and output efficiency values of each unit are calculated

and presented in Table 5. As stated in Remark 3.1, the optimal inputs and outputs required to convert each unit into a Pareto unit are calculated and presented in Table 5. For example, unit C, which is inefficient, becomes a Pareto unit by reducing its inputs to 2 and increasing its outputs to 2.99.

Table 5. Input and Output Efficiency Values

DMU	A	B	C	D	E	F
Input efficiency	1	1	0.5	0.43	1	0.92
Output efficiency	1	1	1.58	1.58	1	1
Input optimal	2	3	2	2	6	5.99
Output optimal	3	4	2.99	2.99	5	5

5-2. Comparison of the Models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), Soltanifar et al. (2024), and Model (10)

1. Since the structure of PPS in model (10) is convex geometry technology, model (10) is more potent in the merger process of units compared to previous models such as Gattoufi et al. (2014), Amin & Ibn Boamah (2020), and Soltanifar et al. (2024).
2. Unlike models of Gattoufi et al. (2014) and Amin & Ibn Boamah (2020), in the proposed model (10), a new unit with the optimal inputs and outputs is obtained.
3. The newly presented model (10) is a non-radial model, and managers can manage the efficiency of all inputs and outputs of the newly obtained unit in the merger process; however, in the models proposed by Gattoufi et al. (2014) and Amin & Ibn Boamah (2020), managers can only manage the efficiency of inputs or outputs of the obtained new unit from the merger as radial. Also, in Soltanifar et al. (2024) model, managers do not intervene in the performance of inputs and outputs of the merged units.
4. If the goal is to obtain a new unit with 100% efficiency, model (10) can identify a new Pareto-efficient unit, whereas in models of Gattoufi et al. (2014) and Amin & Ibn Boamah (2020), a weakly efficient unit may be obtained.
5. In the presented multiplicative model (10), the contribution of each input and output for every merged unit can be determined in the resulting unit. Moreover, the contribution ratio of each input and output is the same in all units and non-zero. In contrast, in models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), and Soltanifar et al. (2024), the contribution of some units in the merger may be zero, and the contribution ratio of inputs and outputs is not the same in all units.
6. Unlike the merger methods of Gattoufi et al. (2014) and Amin & Ibn Boamah (2020), our approach focuses on the feasibility of the presented merger model.

The above points are summarized in the Table 6.

Table 6. Comparisons of Models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), Soltanifar et al. (2024), and Model (10)

Model	PPS technology	Efficient frontier	Model type	Contribution ratio of each indicator	Obtained efficient unit	Feasibility	Target	Type output
Gattoufi et al. (2014)	convex arithmetic	union of segments	radial	uncertain may become zero	weakly efficient	it may become infeasible	predetermined radial efficiency	optimal input or output
Amin & Ibn Boamah (2020)	convex arithmetic	union of segments	radial	uncertain may become zero	weakly efficient	it may become infeasible	predetermined cost efficiency	optimal input or output
Soltanifar et al. (2024)	convex arithmetic	union of segments	CSW	uncertain may become zero	Pareto-efficient	always feasible	predetermined radial efficiency	optimal input and output
Model (10)	convex geometry	curve	non-radial	equal and non-zero	Pareto-efficient	always feasible	predetermined non-radial efficiency	optimal input and output

For example, consider the data in Table 4. We first merge units C and D using the models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), Soltanifar et al. (2024), and model (10). The outcomes are summarized in Table 7.

Table 7. Results for the Merged Units C and D, Based on the Approaches of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), Soltanifar et al. (2024), and Model (10)

Type of model	Model assumptions	Data of the units after the merger
Gattoufi et al. (2014) and Amin & Ibn Boamah (2020)	$\lambda_M = 0, \overline{\theta} = 1, \overline{C} = 1, c_1 = 1$	C (0, 2), D (3, 2), M = C + D = (3, 4)
Soltanifar et al. (2024)	$E_G^* = 1$	C (0, 0), D (2.7, 4), G = C + D = (2.7, 4)
Model (10)	$\bar{\theta}_1 = 1, \bar{\varphi}_1 = 1$	C (1.33, 2), D (1.65, 2), G = C + D = (2.9, 4)

According to the results in Table 8, in the models proposed by Gattoufi et al. (2014) and Amin & Ibn Boamah (2020), the input value of unit C is zero and has no share of the input in the merger. Similarly, in the Soltanifar et al.'s (2024) model, unit C has no share of the input and output in the merger. In contrast, in the proposed model (10), both units C and D have a share of input and output; according to point 4 of subsection 4.3, the share ratio of input of units C and D is equal to 0.33, and the share of output of units C and D is equal to 1. In all four models, the merged unit is efficient; however, based on the convex geometric technology structure of model (10), a more accurate merger is performed.

Now, we merge units C and F, and the corresponding results are presented in Table 8.

Table 8. The Results of the Merger of Units C and F by Gattoufi et al. (2014), Amin & Ibn Boamah (2020), Soltanifar et al. (2024), and model (10)

Type of model	Model assumptions	Data of the units after the merger
Gattoufi et al. (2014)	$\lambda_M = 0, \overline{\theta} = 1$	Infeasible
Amin & Ibn Boamah (2020)	$\overline{C} = 1, c_1 = 1$	C (4, 2), F (7, 5), M = C + F = (11, 7)
Soltanifar et al. (2024)	$E_G^* = 1$	C (0, 0), F (4.7, 7), G = C + F = (4.7, 7)
Model (10)	$\bar{\theta}_1 = 1, \bar{\varphi}_1 = 1$	C (3.34, 2), F (5.84, 5), G = C + F = (9.2, 7)

According to the results of Table 8, the model of Gattoufi et al. (2014) becomes infeasible in merging units C and F. As observed in Table 7, in the model of Soltanifar et al. (2024), the contribution of unit C in both input and output in this merger is zero, which is illogical. However, in the non-radial model (10), based on point 4 of subsection 4.3, the two units C and F participate with a share ratio of 0.83 from the input and a share ratio of 1 from the output in this merger.

6. Implementation

In this section, the proposed models are applied to evaluate and merge a subset of Iranian banks. According to statistics in the World Bank Open Data[1], in most countries, optimal banking service levels are achieved with 10–15 branches per 100,000 adults, whereas Iran has a substantially higher branch density than advanced economies. This overexpansion is seen as harmful to the economy, with critics asserting that resources devoted to opening new branches should instead support production and productivity. They further argue that reducing the number of branches is essential for establishing an efficient and modern banking industry. Furthermore, the development of digital services, such as mobile and internet banking, can improve branch performance and increase their service capacity. Therefore, bank mergers warrant particular attention in Iran.

1. World Bank Open Data is the official data portal of the World Bank, offering open access to a broad range of global development indicators, including information on poverty, health, education, climate change, and more. For further details; see https://data.worldbank.org.

The case study examines 23 Iranian banks, whose data are gathered from the Iran Banking Institute[2]. Like Gattoufi et al. (2014) and Amin and Boamah (2020), in this analysis, two inputs are used: interest expenses (X1) and non-interest expenses (X2). Interest expenses include expenditures for deposits and other borrowings, commissions, doubtful receivables, investment and foreign currency transaction costs, and income taxes. Non-interest expenses consist of salaries, administrative and general expenses, depreciation, and other fees. The outputs used in this study are interest income (Y1)) and non-interest income (Y2), as in Gattoufi et al. (2014) and Amin and Boamah (2020). Interest income includes earnings from loans, deposits, investments, and foreign exchange transactions. Non-interest income consists of service charges, commissions, and other operational income. The corresponding input and output data for the 23 Iranian banks are listed in Table 9.

Table 9. Iranian Banks: Inputs and Outputs (billion Iranian Rials)

Bank number	Input X1	Input X2	Output Y1	Output Y2
1	503716.0	29810.0	252061.0	5786.0
2	129029.0	9383.0	121179.0	22655.0
3	191292.0	19195.0	217119.0	18533.0
4	8490.0	18990.0	19470.0	26692.0
5	204.0	232.0	2348.0	4.0
6	51590.0	11156.0	71051.0	4867.0
7	214919.0	84136.0	276994.0	22292.0
8	37902.0	4094.0	59610.0	2965.0
9	146502.0	67604.0	204944.0	9153.0
10	74666.0	8393.0	60296.0	13240.0
11	446217.0	245631.0	798033.0	54946.0
12	226920.0	25020.0	234405.0	22964.0
13	311806.0	31270.0	411117.0	87156.0
14	27110.0	12099.0	41263.0	10073.0
15	219180.0	80643.0	281755.0	19979.0
16	420385.0	133358.0	464283.0	93103.0
17	102147.0	20302.0	121039.0	23278.0
18	76995.0	5454.0	12412.0	867.0
19	171444.0	24655.0	156707.0	61414.0
20	42448.0	15928.0	60414.0	5518.0
21	352874.0	147788.0	522491.0	57491.0
22	53304.0	7812.0	48384.0	2918.0
23	31209.0	15365.0	39500.0	7952.0

6-1. Comparison of Bank Efficiencies

Using model (1) and the newly presented model (3), we evaluate all the mentioned banks; the obtained efficiency scores from these two models, as well as the inputs and output efficiency scores derived from model (3), are presented in Table 10.

As stated in Section 3, the results presented in Table 4 indicate that the identified banks as Pareto-efficient by model (3) are also Pareto-efficient according to model (1); the efficiency scores of inefficient banks based on model (3) are less than or equal to their efficiency scores by (1). According to model (3), the banks were evaluated in a non-radial manner, and all inefficient banks were identified; whereas, under model (1), the evaluation was conducted radially, and only purely technically inefficient banks were identified. Furthermore, according to Remark 3.1, the input and output efficiency scores of the banks are calculated using model (3) and are presented in Table 4. These values represent the necessary modifications in input utilization and output production for inefficient banks to attain Pareto-efficiency. For example, in Bank 1, the first input value is 503,716, and its efficiency score is 0.92. This score indicates that, for Bank 1 to achieve Pareto-efficiency, the required amount of the first input should be adjusted to $503716^{0.92}$. Similarly, the second output value is 5,786, with an efficiency score of 1.29, meaning that to become Pareto-efficient, Bank 1 needs to increase this output to $5786^{1.29}$.

1. The Iran Banking Institute is an educational and research institution dedicated to developing the skills and knowledge of professionals in the Iranian banking sector; see https://www.ibi.ac.ir.

Table 10. Efficiency Scores of Banks Obtained by Models (1) and (3)

Bank number	Efficiency score of model (1)	Efficiency score of new model (3)	Efficiency score of X1	Efficiency score of X2	Efficiency score of Y1	Efficiency score of Y2
1	0.95	0.84	0.92	1	1	1.29
2	1	1	1	1	1	1
3	0.98	0.95	1	1	1	1.10
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	0.95	0.92	1	1	1	1.16
7	0.96	0.90	0.99	0.91	1	1.12
8	1	1	1	1	1	1
9	0.95	0.86	0.99	0.92	1	1.22
10	0.99	0.98	1	1	1.03	1.02
11	1	1	1	1	1	1
12	0.96	0.94	0.98	1	1	1.10
13	1	1	1	1	1	1
14	0.97	0.96	1	1	1	1.07
15	0.97	0.89	0.99	0.91	1	1.13
16	1	1	1	1	1	1
17	0.97	0.96	0.98	1	1	1.06
18	0.90	0.79	0.96	1	1.18	1.30
19	0.99	0.98	1	1	1.03	1
20	0.94	0.90	0.99	1	1	1.20
21	0.99	0.96	1	0.93	1	1.02
22	0.93	0.91	1	1	1.03	1.18
23	0.94	0.92	0.97	1	1	1.13

6-2. Banks Merging Using Model (10)

Suppose we aim to obtain a new Pareto-efficient bank from merging banks in $E = \{Bank1, Bank5, Bank18\}$. For this purpose, we first obtain $Bank_M = (580915, 35496, 266821, 6657)$ from the total merger of banks in E. By solving model (12), the efficiency score of $Bank_M$ is 0.83, which is not Pareto-efficient and is not a desirable unit for us. Now, to achieve a Pareto-efficient bank by merging the banks in *E*, we apply model (10) and consider $(\bar{\theta}_1 = 1, \bar{\theta}_2 = 1, \bar{\varphi}_1 = 1, \bar{\varphi}_2 = 1)$. By solving the linear model (13), the optimal solution and the optimal value of (10) are determined as $(\alpha_1^* = 0.0854, \alpha_2^* = 0.0188, \beta_1^* = 0, \beta_2^* = 0.2731)$ and $\rho_m^* = 0.83$; in this regard, $Bank_G = (x_{iM}^{1-\alpha_i^*}, y_{rM}^{1+\beta_r^*})$ for $i = 1, 2$ and $r = 1, 2$ is obtained from merging banks in *E*. Moreover, according to advantage 4 of Subsection 4.3, for $k = 1,5,18$,

$$X_{ik}^* = (\frac{1}{x_{iM}})^{\alpha_i^*} x_{ik} \text{ and } y_{rk}^* = y_{rM}^{\beta_r^*} y_{rk}$$

for $i = 1, 2$ and $r = 1, 2$, represent the input and output contributions of $Bank_k$ in this merger, respectively. Table 11 displays the outcomes of this merger.

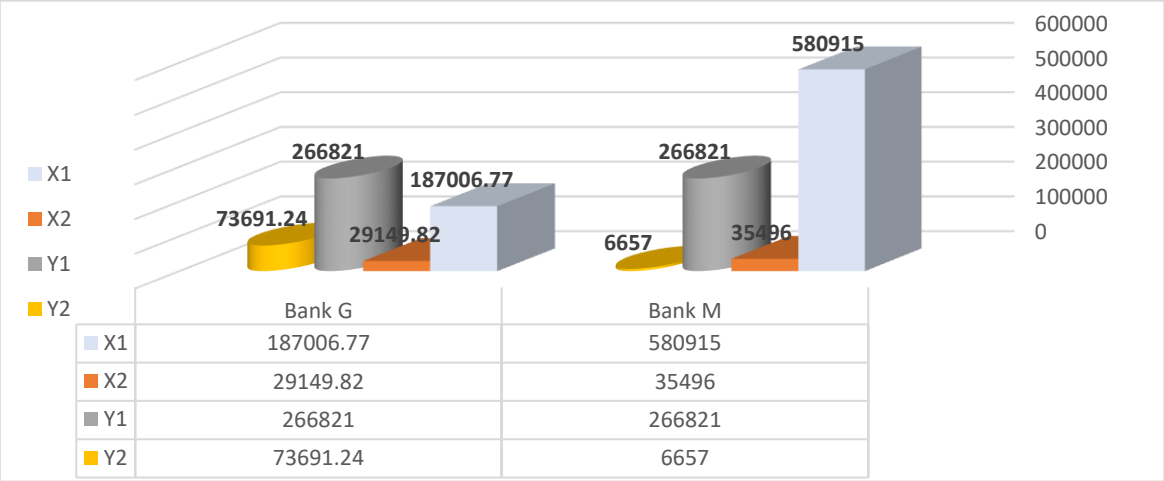
Table 11. Optimal Contribution of Inputs and Outputs of the Merged Banks for $\bar{\theta}_1 = 1, \bar{\theta}_2 = 1, \bar{\varphi}_1 = 1, \bar{\varphi}_2 = 1$

Bank 1 = (503716,29810,252061,5786), Bank 5 = (204,232,2348,4), Bank 18 = (76995,5454,12412,867), $Bank_M$ = (580915, 35496, 266821, 6657)

Optimal inputs contribution of Banks 1, 5, and 18	Optimal outputs contribution of Banks 1, 5, and 18	Efficiency score from model (3)
$X_1^* = (162155.06, 24480.40)$	$Y_1^* = (252061, 64049.50)$	
$X_5^* = (65.67, 190.52)$	$Y_5^* = (2348, 44.27)$	
$X_{18}^* = (24786.04, 4478.90)$	$Y_{18}^* = (12412, 9597.46)$	
New Bank $X_G = (187006.77, 29149.82)$	$Y_G = (266821, 73691.24)$	1

In Table 11, X_k^* and Y_k^* for $k = 1, 5, 18$ are the optimal contributions of inputs and outputs of Banks 1, 5, and 18 in the merger. The share ratio of *i*-th input in all the merged banks of set *E* is a constant value, and similarly, the share ratio of the *r*-th output in all the merged banks in *E* is also a

constant value. As shown in Table 5, these constant values are $\frac{x_{1k}^*}{x_{1k}} = 0.32$, $\frac{x_{2k}^*}{x_{2k}} = 0.82$, $\frac{y_{1k}^*}{y_{1k}} = 1$, and $\frac{y_{2k}^*}{y_{2k}} = 11$ for $k = 1, 5, 18$. As stated in Section 4, the efficiency score of the bank resulting from the merger, i.e., $Bank_G = (X_G, Y_G)$, is one, and it is Pareto-efficient. As shown in Figure 1, $Bank_G$ has lower input values, higher output values, and a higher efficiency score compared to $Bank_M$.



	Bank G	Bank M
X1	187006.77	580915
X2	29149.82	35496
Y1	266821	266821
Y2	73691.24	6657

Fig. 1. Comparing the Inputs and Outputs of Banks *M* and *G*

According to Proposition 4.1, the bounds for the predetermined efficiency are $\bar{\theta}_1 \in [0.92, 1]$, $\bar{\theta}_2 \in [0.98, 1]$, $\bar{\varphi}_1 \in [1, 1]$, and $\bar{\varphi}_2 \in [1, 1.27]$ .In model (10), by setting ($\bar{\theta}_1 = 0.97, \bar{\theta}_2 = 1, \bar{\varphi}_1 = 1, \bar{\varphi}_2 = 1.08$), we derive $Bank_G$ from the merger of the banks in set *E*, with the efficiency value of $\bar{\theta} = \frac{s\sum_{i=1}^{m}\bar{\theta}_i}{m\sum_{r=1}^{s}\bar{\varphi}_r} = 0.95$. The contributions of each input and output from the merged banks are presented in Table 12.

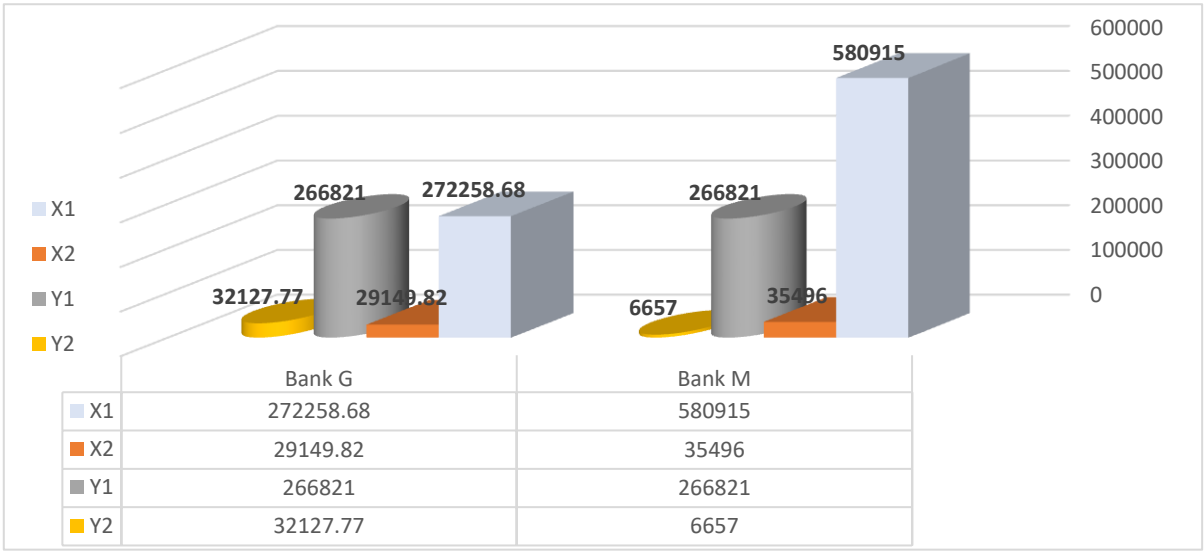
Table 12. Optimal Contribution of Inputs and Outputs of the Merged Banks for $\bar{\theta}_1 = 0.97, \bar{\theta}_2 = 1$, $\bar{\varphi}_1 = 1$ and $\bar{\varphi}_2 = 1.08$

Bank 1 = (503716,29810,252061,5786), Bank 5 = (204,232,2348,4), Bank 18 = (76995,5454,12412,867), $Bank_M = (580915, 35496, 266821, 6657)$,

Optimal inputs contribution of Banks 1, 5, and 18	Optimal outputs contribution of Banks 1, 5, and 18	Efficiency score from model (3)
$X_1^* = (236077.67, 24480.40)$	$Y_1^* = (252061, 27924.19)$	
$X_5^* = (95.60, 190.52)$	$Y_5^* = (2348, 19.30)$	
$X_{18}^* = (36085.41, 4478.90)$	$Y_{18}^* = (12412, 4184.28)$	
New Bank $X_G = (272258.68, 29149.82)$	$Y_G = (266821, 32127.77)$	0.95

In Table 12, X_k^* and Y_k^* for $k = 1, 5, 18$ represent the optimal contributions of inputs and outputs from the merged banks. The result of this merger is $Bank_G$, with the efficiency score of 0.95; based on Remark 3.1, the efficiency scores of inputs x_{1G} and x_{2G} are 0.97 and 1, respectively; also, the efficiency scores of outputs y_{1G} and y_{2G} are 1 and 1.08, respectively. In other words, according to Remark 3.1, $Bank_G$ will become Pareto-efficient after this merger, by reducing its inputs to $(x_{1G}^{0.97}, x_{2G}^{1})$ and increasing its outputs to $(y_{1G}^{1}, y_{2G}^{1.08})$. $Bank_G$, compared to $Bank_M$, has higher efficiency score, and, fewer input values and more output values, as illustrated in Figure 2. In this merger, the share ratios of *i*-th input in all the merged banks in *E* are a constant value; similarly, the share ratios of *r*-th output in all the merged banks in *E* are a constant value. Hence, for $k = 1, 5, 18$, the constant values are $\frac{x_{1k}^*}{x_{1k}} = 0.47$, $\frac{x_{2k}^*}{x_{2k}} = 0.82$, $\frac{y_{1k}^*}{y_{1k}} = 1$, and $\frac{y_{2k}^*}{y_{2k}} = 4.82$. By comparing Tables 11 and 12, and according to the advantage 6 of Subsection 4.3, with the decrease of $\bar{\theta}_1$, the share of the first input of the merged banks has increased, and with the increase of $\bar{\varphi}_2$, the share of the second output of

the merged banks has decreased. Therefore, if bank managers assign lower efficiencies to inputs and outputs, the share of inputs in the merged bank will increase, while the share of outputs will decrease. Consequently, the efficiency of the newly obtained bank will decline, and its value is obtained by $\bar{\theta} = \frac{s\sum_{i=1}^{m}\bar{\theta}_i}{m\sum_{r=1}^{s}\bar{\varphi}_r}$.



	Bank G	Bank M
X1	272258.68	580915
X2	29149.82	35496
Y1	266821	266821
Y2	32127.77	6657

Fig. 2. Comparing the Inputs and Outputs of Banks M and G

6.3. Merging Banks 1, 5, and 18 Using the Models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), and Soltanifar et al. (2024)

To compare the proposed model (10) with the previously introduced models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020) and Soltanifar et al. (2024), we merge Banks 1, 5, and 18 under identical conditions to create an efficient bank. Under the same conditions for all the mentioned models, the parameters are set as follows: $\bar{\theta}_1 = \bar{\theta}_2 = \bar{\varphi}_1 = \bar{\varphi}_2 = 1$ in model (10); $\bar{\theta} = 1$ in Gattoufi et al. (2014); $c_1 = 1, c_2 = 1$ and $\bar{C} = 1$ in Amin & Ibn Boamah (2020); and $E_G^* = 1$ in Soltanifar et al. (2024). The results of this merger using model (10) were previously examined and shown in Table 11, and the results of other models are presented in Table 13.

Table 13. Merging Banks 1, 5, and 18 Using Gattoufi et al. (2014), Amin & Ibn Boamah (2020), and Soltanifar et al. (2024) Models

Type of Model	Inputs after the Merger	Outputs after the Merger
Gattoufi et al. (2014) and Amin & Ibn Boamah (2020)	Bank 1 = (0,0) Bank 5 = (0,0) Bank 18 = (204,232) New Bank G = (204,232)	Bank 1 = (252061, 5786) Bank 5 = (2348, 4) Bank 18 = (12412,867) New Bank G = (266821,6657)
Soltanifar et al. (2024)	Bank 1 = (0,0) Bank 5 = (0,0) Bank 18 = (0,160.0953) New Bank G = (0,160.953)	Bank 1 = (0,0) Bank 5 = (0,0) Bank 18 = (266821,6657) New Bank G = (266821,6657)

To compare the models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), Soltanifar et al. (2024), and model (10), we examine Tables 11 and 13 together. As shown in Table 13, in the models of Gattoufi et al. (2014) and Amin & Ibn Boamah (2020), Banks 1 and 5 have no contribution from their inputs and outputs in the merger, and their values are zero, which is illogical. Moreover, in the model of Soltanifar et al. (2024), the first input of all merged banks is zero; consequently, the first input of the newly formed bank is also zero, indicating that the merger fails. In contrast, Table 11 shows that in model (10), all banks participate in the merger and share in its inputs and outputs. As noted in the description of Table 11, each input share ratio is constant across all merged banks;

likewise, each output share ratio is constant among all merged banks. Furthermore, since model (10) is input and output-oriented, by reducing inputs and increasing outputs of the integrated banks, a new bank with Pareto-efficiency is obtained. By contrast, in the models of Gattoufi et al. (2014), Amin & Ibn Boamah (2020), and Soltanifar et al. (2024), despite the significant reduction in inputs, no increase in the outputs of the new bank is achieved. Given the technology of model (10), the obtained efficiency unit from the merger in Table 11 is more accurate than the derived units from the other models.

6-4. Managerial Implications

The results of this study provide valuable insights for managers and policymakers, specifically in banking. Unlike radial models, the non-radial model (3) allows decision-makers to assess the efficiency of each input and output separately, rather than assigning a uniform efficiency across all variables. This feature offers several practical advantages.

First, identifying input efficiencies enables bank managers to determine which resources should be reduced or reallocated after assessment. For instance, if labor or branch-related expenses are found inefficient, managers can implement restructuring strategies such as downsizing redundant branches, optimizing staffing, or investing in technologies to reduce operational costs.

Second, the ability to set explicit efficiency targets for outputs provides managers with a tool to strategically enhance revenue-generating activities. For example, banks can focus on improving lending portfolio efficiency, expanding digital banking services, or upgrading customer service channels.

Third, this framework is especially relevant for the Iranian banking sector, where regulatory pressures, limited resources, and intense competition necessitate tighter control over inputs and outputs. The flexibility of non-radial efficiency management allows managers to adapt strategies to local priorities such as improving capital adequacy, reducing non-performing loans, or strengthening liquidity without imposing uniform reductions or increases across all dimensions.

From a policy perspective, managers can also employ model (10), derived from model (3), to merge the inefficient banks more precisely. By assigning predetermined efficiency targets for inputs and outputs, policymakers can design mergers that reduce input consumption and increase output production, thereby not only enhancing efficiency but also ensuring the balanced input and output shares among the merged banks. This contributes to enhancing system-wide stability and resilience within the banking sector.

By explicitly controlling the efficiency of inputs and outputs, policymakers can guide mergers toward outcomes that additionally achieve a bank with certain efficiency, align with broader systemic objectives, such as maintaining financial stability, improving risk management, and safeguarding customer service quality. For instance, regulators may set input efficiency targets to control capital adequacy or reduce redundant operational expenses, while simultaneously setting output efficiency targets to improve loan quality or expand digital banking services. Consequently, model (10) provides policymakers with a structured framework to design mergers that support regulatory priorities and contribute to the long-term sustainability of the financial sector.

In sum, the non-radial approaches (3) and (10) translate efficiency analysis into actionable guidance: optimizing resource allocation, enhancing service delivery, and aligning performance improvements with both market and regulatory priorities. This transforms efficiency measurement from a purely diagnostic function into a strategic decision-making tool in the banking industry. All computations in this study were performed using MATLAB 2017.

7. Discussion

As stated in Theorem 3.2 of Section 3, the proposed model (3), leveraging the technology of convex geometry, can identify Pareto-efficient and all inefficient units more accurately than traditional models such as CCR and BCC. Unlike the radial model (1), which detects only technically inefficient and weakly efficient units (Cooper et al., 2007), model (3) identifies all inefficient units and provides efficiency scores more accurately (see Proposition 3.4). This capability enables managers to target inefficient units for potential mergers.

Accordingly, model (10), which is derived from model (3), facilitates precise merger. Unlike previous merger models, the efficiency frontier of model (10) is a curve, hence allowing a more realistic and accurate efficiency unit to be obtained from the merger. Furthermore, since model (10) is non-radial, it enables managers to monitor both the efficiency score of the new unit and the contribution of the merged units by explicitly controlling input and output efficiencies; by reducing the input and output efficiency scores, the merged units have a larger share of inputs and a smaller share of outputs in the merger, but this leads to a lower efficiency score for the unit resulting from the merger. Moreover, since model (10) is input and output-oriented, it simultaneously determines the shares of both inputs and outputs for the merged units.

As demonstrated in Subsection 4.4, the nonlinear structures of models (3) and (10) can be reformulated as the linear models (12) and (13), respectively. These linear forms avoid computational difficulties and can be efficiently solved using standard linear programming software. Moreover, the computational burden of model (10) is actually lower than that of models such as Amin & Ibn Boamah (2020) and Soltanifar et al. (2024). In the Amin & Ibn Boamah (2020) model (model (7)), the cost efficiency model (model (8)) must also be solved. Also, in the Soltanifar et al. (2024) model (model (9)), the CCR efficiency of all non-merging units must be computed, requiring the solution of a large number of linear problems.

8. Conclusion

Regarding the importance of multiplicative models in DEA, this study introduces new multiplicative approaches to evaluate and merge decision-making units within the production possibility set using convex geometry and variable returns to scale technology. The first model evaluates units in a non-radial, input-output-oriented manner and, unlike multiplicative radial models, can identify all Pareto-efficient and inefficient units.

The second model generates a new unit from merging units with optimal inputs and outputs based on defined efficiency targets. In this process, the contributions of the inputs and outputs from the integrated units can be determined, such that the contribution ratio of each input and output is equal across all units and non-zero. In contrast, previous methods for merging units within the production possibility set using convex arithmetic technology may assign zero contribution to some units in the merger.

Furthermore, since the proposed models are non-radial, managers can control the efficiency of all inputs and outputs of the newly formed unit during the merger. In contrast, prior models (e.g., Gattoufi et al. (2014) and Amin & Ibn Boamah (2020)) only allow managers to manage either inputs or outputs radially.

For future studies, the proposed models can be adapted to evaluate and merge units with network structures, and to incorporate negative, interval, or fuzzy data. Additionally, the model can be employed to rank all units.

Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

Al-Sharkas, A., Hassan, M., & Lawrence, S. (2008). The impact of mergers and acquisitions on the efficiency of the US banking industry: Further evidence. *Journal of Business Finance & Accounting, 35*(1&2), 50-70. https://doi.org/10.1111/j.1468-5957.2007.02059.x

Amin, G. R., Emrouznejad, A. (2007). Inverse linear programming in DEA. *International Journal of Operational Research, 4*(2), 105-109.

Amin, G. R., Emrouznejad, A., & Gattoufi, S. (2017a). Modelling generalized firms' restructuring using inverse DEA. *Journal of Productivity Analysis, 48*(1), 51-61. https://doi.org/10.1007/s11123-017-0501-y

Amin, G. R., Emrouznejad, A., & Gattoufi, S. (2017b). Minor and major consolidations in inverse DEA: Definition and determination. *Computers & Industrial Engineering, 103*, 193-200. https://doi.org/10.1016/j.cie.2016.11.029

Amin, G. R., Ibn Boamah, M. A. (2020). New inverse DEA cost efficiency model for estimating potential merger gains: A case of Canadian banks. *Annals of Operations Research, 295*, 21-36. https://doi.org/10.1007/s10479-020-03667-9

Banker, R. D., Charnes, R. F., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science, 30*(9), 1078-92. https://doi.org/10.1287/mnsc.30.9.1078

Bogetoft, P., Wang, D. (2005). Estimating the potential gains from mergers. *Journal of Productivity Analysis, 23*, 145-171. https://doi.org/10.1007/s11123-005-1326-7

Bolós, V. J., Benítez R., & Coll-Serrano, V. (2023). Continuous Models Combining Slacks-Based Measures of Efficiency and Super-efficiency. *Central European Journal of Operations Research, 31*, 363–391. https://doi. org/10.1007/s10100-022-00813-5

Charnes, A., Cooper, W. W., Seiford, L., & Stutz, J. (1982). A multiplicative model for efficiency analysis. *Socio-Economic Planning Sciences, 16*(5), 223-224. https://doi.org/10.1016/0038-0121(82)90029-5

Charnes, A., Cooper, W. W. (1962). Programming with linear fractional functionals. *Naval Research Logistics Quarterly, 9*(3-4), 181-186. https://doi.org/10.1002/nav.3800090303

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision-making units. *European Journal of Operational Research, 2*(6), 429-444. https://doi.org/10.1016/0377-2217(78)90138-8

Cherian, J. A., Jarrow, R. A. (1995). Market manipulation. S. G. Henderson, & B. L. Nelson (Eds.), *Handbooks in operations research and management science, 9*, 611-630. https://doi.org/10.1016/S0927-0507(05)80064-7

Cooper, W. W., Seiford, L. M., & Tone, K. (2007). *Data envelopment analysis: a comprehensive text with models, applications, references and DEA-solver software* (Vol. 2, p. 489). springer. https://doi.org/10.1007/978-0-387-45283-8

Emrouznejad, A., Yang, G., & Amin, G. R. (2019). A novel inverse DEA model with application to allocate the CO2 emissions quota to different regions in Chinese manufacturing industries. *Journal of the Operational Research Society, 70*(7), 1079-1090. https://doi.org/10.1080/01605682.2018.1489344

Fakharzadeh Jahromi, A. R., & Rostamzadeh, H. (2026). New non-radial models for merger and acquisition to achieve strong efficiency and MPSS without changing in the efficient frontier. *Journal of Mathematical Modeling, 14*(1), 35-55. https://doi.org/10.22124/jmm.2025.30974.2781

Fang, Z., Gui, W., Han, Z., & Lan. L. (2024). The efficiency evaluation and influencing factor analysis of regional green innovation: A refined dynamic network slacks-based measure approach. *Kybernetes, 53*(6), 2153–2193. https://doi.org/10.1108/K-03-2022-0420

Farrell, M. J. (1957). The measurement of productive efficiency. *Journal of the Royal Statistical Society, 120*, 253-281.

Gattoufi, S., Amin, GR., & Emrouznejad, A. (2014). A new inverse DEA method for merging banks. *IMA Journal of Management Mathematics, 25*, 73-87. https://doi.org/10.1093/imaman/dps027

Gaughan, P. A. (2013). *Maximizing corporate value through mergers and acquisitions: A strategic growth guide*. John Wiley & Sons.

Hannan, T., Pilloff, S. (2009). Acquisition targets and motives in the banking industry. *Journal of Money, Credit and Banking, 41*(6), 1167-1187. https://doi.org/10.1111/j.1538-4616.2009.00254.x

Healy, P. M., Palepu, K. G., & Ruback, R. S. (1992). Does corporate performance improve after mergers? *Journal of Financial Economics, 31*(2), 135-175.

Kalantary, M., & Saen, R. F. (2019). Assessing sustainability of supply chains: An inverse network dynamic DEA model. *Computers & Industrial Engineering, 135*, 1224-1238. https://doi.org/10.1016/j.cie.2019.06.022

Kao, C. (2023). *Network data envelopment analysis*. Springer International Publishing.

Khoveyni, M., &Eslami, R. (2022). Merging two-stage series network structures: A DEA-based approach. *OR Spectrum, 44*, 273-302. https://doi.org/10.1007/s00291-021-00653-w

Lozano, S., &Contreras, L. (2022). Size efficiency, splits and merger gains, and centralized resource reallocation of Spanish public universities. *Socio-Economic Planning Sciences, 81*, 101-190. https://doi.org/10.1016/j.seps.2021.101190

Lozano, S., Villa, G. (2010). DEA-based pre-merger planning tool. *The Journal of the Operational Research Society, 61*(10), 1485-1497. https://doi.org/10.1057/jors.2010.4

Pendharkar, P. C. (2002). A potential use of data envelopment analysis for the inverse classification problem. *Omega, 30*, 243-248. https://doi.org/10.1016/S0305-0483(02)00030-0

Pyra, M., &Siedlecka, A. (2022). Potential benefits from university mergers: an attempt to evaluate future efficiency levels with a selected DEA model. *European Research Studies Journal, 25*(2), 665-683.

Soltanifar, M., Ghiyasi, M., Emrouznejad, A., & Sharafi, H. (2024). A novel model for merger analysis and target setting: A CSW-Inverse DEA approach. *Expert Systems with Applications, 249*(Part A), 123-326. https://doi.org/10.1016/j.eswa.2024.123326

Tone, K. (1996). A simple characterization of returns to scale in DEA. *Journal of the Operations Research Society of Japan, 39*, 604-613.

Wei, Q., Zhang, J., & Zhang, X. (2000). An inverse DEA model for inputs/outputs estimate. *European Research Studies Journal, 121*, 151-163.

Xiao, S., Li, Y., Emrouznejad, A., Xie, J., & Liang, L. (2017). Estimation of potential gains from bank mergers: A novel two-stage cost efficiency DEA model. *Journal of the Operational Research Society, 68*(9), 1045-1055. https://doi.org/10.1057/s41274-016-0106-2

Xie, Y., Lu, W., & Wang, Y. (2019). Evaluating the efficiency of post-M&A banks with undesirable outputs: A dynamic network DEA approach. *European Journal of Operational Research, 278*(1), 228-239.