Iranian Journal of Management Studies (IJMS) *http://ijms.ut.ac.ir/*
Vol. 9, No. 4, Autumn 2016 *Print ISSN: 2008-7055*
pp. 783-793 *Online ISSN: 2345-3745*

Application of *n*-distance balanced graphs in distributing management and finding optimal logistical hubs

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(Received: July 31, 2016; Revised: October 29, 2016; Accepted: November 3, 2016)

Abstract

Optimization and reduction of costs in management of distribution and transportation of commodity are one of the main goals of many organizations. Using suitable models in supply chain in order to increase efficiency and appropriate location for support centers in logistical networks is highly important for planners and managers. Graph modeling can be used to analyze these problems and many others such as the management of municipal services and traffic control. To achieve these goals, we suggest some models based on structure of distance balanced graphs, and n-distance balanced graphs. These graphs can be considered as a model in communication networks in order to avoid additional costs and maintain balance in networks.

Keywords

Logistics network, *n*-distance balanced graph, Opportunity index, Optimization of network, Supply chain management.

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Introduction

In order to provide an optimal model for solving practical problems, many researchers are interested in graph theory. We know that graphs serve as mathematical tools to analyze many distinguished and concrete real-world problems successfully. It has also series of novel applications in many branches of science, for example see (Creaco et al., 2014; Likaj et al., 2013; Harary, 2013). The significance of graph modeling for natural phenomena makes researchers to classify graphs based on their properties. For instance, a new class of graphs so-called distance balanced graphs introduced by Jerebic et al. (2008) has appropriate structure for distributing management and finding optimal logistical hubs and investigated by many scientists and researchers (for example see Alaeiyan et al., 2016).

Using suitable models in supplying chain in order to increase efficiency and reduce costs (Wagner & Neshat, 2010) and appropriate location for support centers in logistical networks (Chang & Lin, 2015) are an important and noticeable problem for managers and planners (Zhong, 2014). Graph modeling methodology can be used for these problems, and many other similar problems such as the traffic problems in a city (Dave & Jhala, 2014), and the management of municipal services (Maity et al., 2015).

We know that equilibrium in communication networks and its preservation in inevitable events is so important and vital (Tamura et al., 2011). In network optimization, we sometimes need to remove some links to prevent additional costs. To achieve this goal, the concept of distance balanced graphs is generalized and the concept of n-distance balanced graph is introduced. It seems that, this new concept has more advantages than previous concept. For example, in this new concept for some values n, deleting some links cannot destroy distance balanced property.

A graph G as a representation of network is made up of nodes (or vertices) which are connected by links. All graphs considered in this paper are simple connected graph without loops and parallel link. The set of all nodes and the set of all links of G is denoted by $V(G)$ and

$E(G)$, respectively. The number of nodes is called order of G and the number of links is called order of G. Suppose x and y are two nodes of G. The distance $d_G(x,y)$ between these nodes is the minimum length of all paths between them. Whenever G is clear, we will write $d(u,v)$ instead of $d_G(u,v)$. The diameter of G is the maximum distance between all nodes.

If xy is a link of G then three sets W_{xy}, W_{yx} and ${}_xW_y$ where:

$$W_{xy} = \{u \in V(G): d_G(x,u) < d_G(u,y)\},$$
$$W_{yx} = \{u \in V(G): d_G(y,u) < d_G(u,x)\},$$
$${}_xW_y = \{u \in V(G): d_G(x,u) = d_G(u,y)\},$$

form a partition for $V(G)$, i.e., the union of them is equal to $V(G)$ and there is no node in the intersection of these sets. The subscript G can also be omitted if no ambiguity arises. Jerebic et al. (2008) introduced a class of graphs based on these sets. A graph G is said to be distance balanced if the relation $|W_{xy}| = |W_{yx}|$ occurs for every link $e = xy$ of G.

Many famous family of graphs are distance balanced as hyper-cube, cycle, complete and generalized Peterson graphs. In this study, the researchers suggest some models based on structure of distance balanced graphs. Also, the concept of n-distance balanced graphs as an extension of distance balanced graphs for use in modeling is given. This extension has more advantages than the original definition. For example, removal link of n-distance balanced graphs for some values n, do not destroy its property. Therefore, using of distance balanced structures help managers and planners to design suitable networks for optimization and reduction of costs. Our notations used in this article, are standard and taken mainly from the standard books on these topics.

Influence of link removal on *n*-distance balanced property

Distance balanced graphs can be generalized to special cases of graphs that named n-distance balanced graph where 1-distance balanced property is coincident to distance balanced property. Let W_{xny} be the set of all nodes of G closer to x than y.

Definition 1. A simple connected graph G is said to be n-distance balanced graph if for all nodes u and v at distance n holds the equality $|W_{unv}| = |W_{vnu}|$.

The concept of 1-distance balanced property is coinciding to distance balanced property and therefore a graph with n-distance balanced property does not need to be distance balanced. For instance, the complete bipartite graph $K_{m,n}$ and wheel graph W_n are 2-distance balanced, but not distance balanced. These graphs have diameter two. The friendship graph F_n with $2n$+1 nodes and $3n$ links is also non-distance balanced graph; however, has 2-distance balanced property.

Removing link from a graph does not keep distance balanced property.

Theorem 1. (Jerebic, 2008) Let G be a distance balanced graph with at least two links and e is a link of it. Then obtained graph by removing link e of G is not distance balanced.

The above theorem does not hold for 2-distance balanced graphs. In other words, in these graphs there are some certain links in which removing these links, do not destroy 2-distance balanced property. This expressed property for these graphs is a preference rather than distance balanced graphs.

Example 1. Consider cycle graph C_3, complete graph K_4 and diamond graph D and certain links are depicted in Figure 1. These graphs are 2-distance balanced and removing certain links do not destroy 2-distance balanced property.

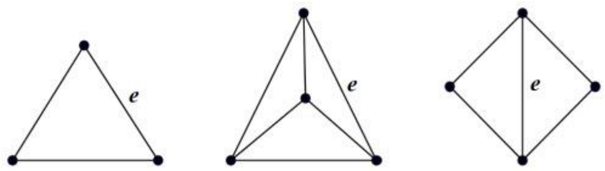

Fig. 1. Cyclic graph C_3, complete graph K_4 and Diamond graph D

It seems that the definition of generalized distance balanced has more advantages than distance balanced. For instance, link removal of some these graphs does not destroy its property, and in the modeling of social networks in order to avoid additional costs while some links are removed, conservation of the balanced property can be considered. Therefore if we choose appropriate links and remove them, at the same time we achieve our aims, and also cover the equilibrium of network.

Distance balanced graphs and equal opportunity networks

For a node u of simple connected graph G, the total distance $td(u)$ is defined as the summation of all distances between u and all nodes of G, i.e. $td(u) = \sum_{v \in V(G)} d_G(u,v)$ (Brandes & Erlebach, 2005).

Example 2. The total distance of every node of cyclic graph C_n is equal to $n^2/4$ where n is even integer and otherwise is $(n^2\text{-}1)/4$.

In distance balanced graphs, all nodes have the same total distance. This is necessary and sufficient condition for these graphs. In other words, for recognizing distance balanced graphs using the concept of total distance is one of the suitable methods.

A new concept related to the total distance is opportunity index and introduced by Balakrishnan et al. (2014). When we partition nodes of network into two equal sets, these sets may have a very different structure. We are especially interested to consider a network having the opportunity index equal to 0. Networks with this property have symmetry in metric measure in graph theory.

Given a graph G and suppose X is an arbitrary subset of G nodes. The relative Wiener index of X in G is defines as the summation of distances between all nodes in X, i.e. $W_X(G) = \sum_{u,v} d_G(u,v)$.

Example 3. Consider complete bipartite $K_{m,n}$ with two partition A and B where $|A| = m$, $|B| = n$ and $V(H_{m,n}) = A \cup B$. The relative wiener index of A and B is equal to $2m(m\text{-}1)$ and $2n(n\text{-}1)$ respectively.

Definition 2. Let G be a graph in which the number of nodes is even number $2n$. Let V_1 and V_2 be two same size subsets of nodes of G such that $V_1 \cup V_2 = V(G)$. This implies $|V_1| = |V_2| = n$ and $V_1 \cap V_2 = \emptyset$, and we say that $\{V_1,V_2\}$ is a half partition of G. The opportunity index of a graph G is defined as:

$opp(G)= \max\{|W_{V_1}(G) - W_{V_2}(G)| : \{V_1,V_2\}$ is a half partition of $G\}$.

Example 4. Consider complete graph K_{2n} on $2n$ nodes. Then $opp(K_{2n}) = 0$. Now attach a leaf to each node. Next, opportunity index of this graph equals to $2n(2n$-$1)$.

This example shows that there are graphs G such that $opp(G)$ is arbitrary large. Therefore, it is interested to investigate graphs with opportunity index equal to 0.

Definition 3. A graph G of order even is said to be opportunity graph if $opp(G) = 0$.

Example 5. The complete graph K_{2n}, complete bipartite graph $K_{n,n}$ and cycle graph C_n are opportunity graph.

Graphs in example 5 are distance balanced and in general, Balakirishnan and his coauthors gave a characterization of distance balanced graphs.

Theorem 3. (Balakrishnan et al., 2014) A graph G is opportunity graph if and only if it is distance balanced graph of order even.

Now, we point an important problem in graph theory named Wiener game, which establishes an important relationship with distance balanced graphs. This game runs on connected graph of even order. Nodes selected by two players say A and B at the same time. Player A starts, and then players choose nodes in order until all nodes are selected. Let V_A and V_B be selected nodes by A and B, respectively. Since the order of G is even, we have $|V_A| = |V_B|$. The aim of players is to minimize the quantities $W_A(G) = \sum_{u,v\in V_A} d_G(u,v)$ and $W_B(G) = \sum_{u,v\in V_B} d_G(u,v)$. With this assumption both players were playing optimally and accurately. If $W_A(G) < W_B(G)$ (or $W_B(G) < W_A(G)$) then A wins the game (or B wins) and otherwise will draw. This game has a win if the value $|W_A(G) - W_B(G)|$ is positive. Therefore, players want to maximize this value and the study of upper and lower bounds of $|W_A(G) - W_B(G)|$ will help players. Opportunity index is an upper bound for it, and finding the lower bound for it is needed. In this game, investigating graphs with positive upper and lower bound are very interesting, because in such graphs the game can never be a draw. Vice versa, we can study some conditions which lead to draw.

Theorem 4. If G is a distance balanced graph of even order, then Wiener game on G leads to draw, regardless of the strategy used by either of the player (Balakrishnan et al., 2014).

From this point of view, in opportunity networks, Wiener game is a draw.

Optimized of computations using distance balanced graphs measurement and comparisons of networks

Total distance index of network which is defined as td(network) = $\sum_{u,v\in V} d(u,v)$ is an important factor in comparing two designs of a network. Cost-effective ratio index is other measurement scale that defined as CER(network) = (total distance)/(cost) (Dixon & Lundeen, 2004). This index can be obtained by computing the total distance of network and related costs. The performance of this index returns to determine the advantages strategy for given budget. The constancy of total distance of all nodes in distance balanced graphs leads to optimization of computations in order to achieve the rate of cost-effective (Vanhook, 2007).

Example 6. Consider m-dimensional hypercube Q_m with 2^m nodes and $m2^{m-1}$ links whose nodes may be described as the binary strings of length m such that two nodes link if and only if the corresponding binary strings differ in a single bit. This graph is distance balanced and so the total distance of arbitrary node u is equal to $td(u) = m2^{m-1}$.

The hypercube Q_m is shown in Figure 2 for $m = 1, 2, 3$ and the total distance of every nodes in Q_1, Q_2 and Q_3 is equal 1, 4 and 12, respectively.

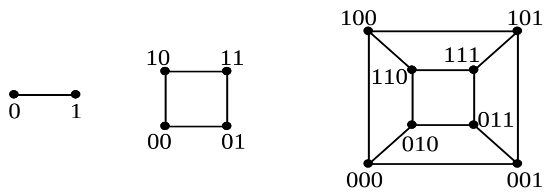

Fig. 2. Hypercube graphs

Therefore, with the assumption constancy of production cost of every link, this index is equal to $CER(Q_m) = m\times 2^{2m-1}/c$ where c is the cost of construction of one link.

Theorem 5. Let G be a graph of a network on n nodes. If G is distance balanced and c is primary budget of construction cost of an arbitrary link between two nodes in optimum way, then $CER(G) = n \times td(u)/c$ where u is an arbitrary node of G.

The above theorem shows that computation of such important index in distance balanced graphs needs to only total distance of one node. It shows the significant reduction of computational cost in distance balanced structures.

In what follows, it indicates another perspective of distance balanced structures. Supposedly, two electronic products distribution companies *A* and *B* want to distribute their products between buyers including sellers of electrical appliances in the whole city. In order to maintain the balance of the supply of products among all seal terminals (PoS) and fair service to citizens, these companies intend to distribute their products among all branches fairly. The aim of each of companies is minimizing the losses and damage caused during transportation (Ruchansky et al., 2015).

If we consider seal terminals as nodes of a graph and transmission between terminals as links, then due to the purpose of companies, the summation of all distances between nodes must be minimized. The structure of seal terminals, during construction must be such that the companies have no losses. The structure of distance balanced graphs achieves this goal, because in these structures all elections are fair and cause to share the damages of transmissions between companies.

In the following a numerical example is given.

Example 7. Consider the graph G as shown in Figure 3. Let VP and VQ be the collection of PoS such that companies P and Q intend to distribute their products to these terminals, respectively. Obviously, the number of elements of two collections must have the same size. Now for a distribution as shown in Figure 2, we have:

$$W_P = \sum_{u,v \in VP} d(u,v)$$
$$= td(P_1) + td(P_2)|(P\text{-}P_1) + td(P_3)|(P\text{-}(P_1 \cup P_2)) + \ldots + td(P_9)|(P\text{-}(P_1 \cup \ldots \cup P_8))$$
$$= 22 + 18 + 19 + 9 + 10 + 4 + 4 + 1 + 0 = 87.$$

$$W_Q = \sum_{u,v \in VQ} d(u,v)$$
$$= td(Q_1) + td(Q_2)|(Q\text{-}Q_1) + td(Q_3)|(Q\text{-}(Q_1 \cup Q_2)) + \ldots + td(Q_9)|(Q\text{-}(Q_1 \cup \ldots \cup Q_8))$$
$$= 18 + 22 + 14 + 15 + 6 + 6 + 4 + 2 + 0 = 87.$$

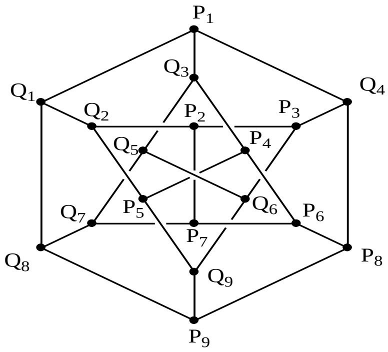

Fig. 3. The distance balanced graph *G*

This example confirms that any selection of seal branches leads to similar and fair damage for two companies. Therefore, it is enough to obtain minimum value of W_p (or similarly for W_Q) for two companies have minimal losses. We remained that the median node of a graph is a node with minimum total distance, and then we must find these nodes in graph for this aim.

Conclusion

In this paper, it is shown how graph theory can be used to model some practical problems such as the distribution and transportation problems, and we apply distance balanced graphs and *n*-distance balanced graphs as a modeling for management of these problems.

In the modeling of transportation problem to weighted graphs, the balanced property is very important. For instance, consider post offices in a city who are obligated to distribute mails. The purpose of transportation is to send a parcel post with the lowest costs. According to the Wiener game, post offices should be built such that they have same loss and damage due to the urban transport. This purpose is achievable when we use distance balanced graphs as the model of this problem.

In a distance balanced structure, if two post transportation offices want to cover same number of different areas in a city to transport postal packets and the number of these areas be even number, then these two post offices can be built anywhere in the city and have the same cost in transportation.

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