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A General Dynamic Function for the Basal Area of Individual Trees Derived from a Production Theoretically Motivated Autonomous Differential Equation

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Abstract

The management of forests may be motivated from production economic and environmental perspectives. The dynamically changing properties of trees affect environmental objectives and values of trees as raw material in the construction sector and in the energy sector. In order to optimize the management of forests, it is necessary to have access to reliable functions that predict how trees develop over time. One central property of a tree is the basal area, the area of the stem segment 1.3 meters above ground. In this paper, a general dynamic function for the basal area of individual trees has been developed from a production theoretically motivated autonomous differential equation. A closed form solution is derived and analyzed. Several examples of recent application of this function in Iran and Sweden are reported.

Keywords

Dynamic function, differential equation, basal area, forest growth.

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Introduction

The forests represent a large part of what is often considered as the environment.

Kordshouli et al. (2015) write:

"Nowadays, paying attention to the environmental needs and desires of consumers and trying to satisfy them by designing marketing activities in an environmentally friendly way, is the best way to increase their satisfaction and finally, gain competitive advantage."

In order to understand how the environment is changing over time, it is important to understand the dynamic development of trees. With such understanding, it is often possible to adapt the management of forests to environmental needs and desires of consumers. Trees are however, not only important as components of the environment. They can sometimes be used to produce building materials, paper, energy and much more.

Nazari et al. (2016) try to predict long term energy demand. They write:

"The objectives of the study are: (1) Selecting the appropriate scenario to predict the energy demands of residential and commercial sectors in Iran; (2) prediction of the annual energy demand of residential and commercial sectors in Iran up to 2032."

Energy demand is a function of price. Energy prices have, during the latest decades, shown considerable variation. Some energy sources are exhaustable and some are not. The trees in the forests represent a potential sustainable source of bioenergy supply. In order to understand how big this resource is and how it changes and can be managed over time, we need reliable growth functions.

Orand et al. (2015) optimize inventory control. They write:

"In this study, we considered an inflationary inventory control model under non-deterministic conditions. We assumed the inflation rate as a normal distribution, with any arbitrary Probability Density Function (PDF). The objective function was to minimize the total discount cost of the inventory system. We used two methods to solve this problem. One was the classic numerical approach which turned out to be prohibitively difficult. The other was a proposed combination method which used Simpson approximation and Particle swarm optimization (PSO). To illustrate the theoretical results, we have provided numerical examples."

From a management point of view, it is important to be aware that also the forest, and more concretely, the trees in the forest, represent an inventory. This inventory is often very costly to keep at a large level. The optimal way to manage a forest with trees of different sizes, developing over time as a function of management activities of different kinds, is impossible to determine without reliable growth functions. Furthermore, as Orand et al. (2015) report, some calculation methods are very difficult to use. One of the central difficulties when forest management decisions should be optimized, is that multi period calculations with discrete time models in high resolution are extremely time consuming. For this reason, it is very valuable to have access to closed form solutions to reliable and robust differential equations that describe the growth of trees. With such closed form solutions, good predictions of the future properties of trees can be instantly obtained, without large numbers of partial calculations in discrete time steps.

The theoretical gap can be formulated this way: Until now, dynamic forest management optimization has usually been studied via comparative statics analysis based on general function analysis (a), discrete time numerical optimization (b), or continuous time optimal control theory based on the logistic law of population growth (c) as described by Braun (1983). In the first case (a), theoretically correct principles have been derived but closed form solutions and numerical approximations have not been obtained. In the second case (b), approximate solutions to the optimization problems have been derived but closed form solutions could not be derived. Furthermore, the execution times have been considerable. In the third case (c), closed form solutions have been obtained, but the problems had to be adjusted to less relevant versions of the original problems. Now, with the new differential equation described and analyzed in this paper, closed form solutions can be derived to the original optimization problems. Hence the degree or relevance and the precision are improved and the computation time is strongly reduced.

Lohmander (2018a) and Lohmander (2018b) include several types of management optimization models where dynamic equations of tree properties are needed.

Individual trees have several properties that may be of interest to predict. One central property is *the basal area*, the area of the stem segment 1.3 meters above ground.

In this paper, a general dynamic function for the basal area of individual trees will be derived from a production theoretically motivated autonomous differential equation. In the following pages, a closed form solution will be developed and analyzed.

Finally, some examples of recent application of this function will be reported.

The mathematical methods utilized in this paper are well presented by Braun (1983) and Simmons (1972). Now, fundamental biological production theory will suggest a relevant differential equation. We start with a small tree. Consider a stem segment, of height H, of the tree. The stem segment is cylindrical with diameter D_1. The leaves cover a cylinder with diameter D_2. $D_2 = \gamma D_1, \gamma > 1$. The sun light reaches the tree from the side. V is the volume of the stem segment. x is the basal area, $x = (\pi / 4)D_1^2$. $V = Hx$. Volume increment is proportional to the photosynthesis level, P, which in turn is proportional to the sun light projection area on the leaves, A, $A = HD_2$. We may conclude that:

$\frac{dV}{dt} = H\frac{dx}{dt} \propto P \propto A \propto D_2 \propto D_1 \propto \sqrt{x}$. Hence, $\frac{dx}{dt} \propto \sqrt{x}$ or $\frac{dx}{dt} = a\sqrt{x}, a > 0$. As the size of the tree increases, the production efficiency declines. Furthermore, the value of γ is often lower for large trees than for small trees. A relevant function considering this is:

$$\frac{dx}{dt} = a\sqrt{x}\left(1 - cx\right) \quad a > 0, c > 0, 0 < x < c^{-1}.$$

Mathematical Model Development and Analysis

$$\frac{dx}{dt} = ax^{0.5} - bx^{1.5} \qquad a > 0, b > 0, 0 < x < \frac{a}{b}$$

$$\frac{dx}{dt} = a\sqrt{x}\left(1 - cx\right) \quad , \quad c = \frac{b}{a} > 0, 0 < x < c^{-1}$$

$\frac{1}{\sqrt{x}\left(1-cx\right)}dx = a\,dt$. Integration gives $\int \frac{1}{\sqrt{x}\left(1-cx\right)}dx = \int a\,dt + k_0$

. $\frac{\ln\left(\sqrt{c}\sqrt{x}+1\right) - \ln\left(\sqrt{c}\sqrt{x}-1\right)}{\sqrt{c}} = at + k_0$. Let us investigate the left hand side, called Z.

$$\frac{dZ}{dx} = \frac{1}{\sqrt{c}\left(\sqrt{c}\sqrt{x}+1\right)}\left(\frac{\sqrt{c}}{2\sqrt{x}}\right) - \frac{1}{\sqrt{c}\left(\sqrt{c}\sqrt{x}-1\right)}\left(\frac{\sqrt{c}}{2\sqrt{x}}\right)$$

$$\frac{dZ}{dx} = \frac{1}{2\sqrt{x}\left(\sqrt{c}\sqrt{x}+1\right)} - \frac{1}{2\sqrt{x}\left(\sqrt{c}\sqrt{x}-1\right)}.$$

Then,

$$\frac{dZ}{dx} = \frac{\left(\sqrt{c}\sqrt{x}-1\right) - \left(\sqrt{c}\sqrt{x}+1\right)}{2\sqrt{x}\left(\sqrt{c}\sqrt{x}+1\right)\left(\sqrt{c}\sqrt{x}-1\right)}$$

$\frac{dZ}{dx} = \frac{1}{\sqrt{x}\left(1-cx\right)}$. This confirms that the integration was correct.

$\frac{\ln\left(\frac{\sqrt{c}\sqrt{x}+1}{\sqrt{c}\sqrt{x}-1}\right)}{\sqrt{c}} = at + k_0$ which leads to $\ln\left(\frac{\sqrt{c}\sqrt{x}+1}{\sqrt{c}\sqrt{x}-1}\right) = \sqrt{c}at + k_1 \quad k_1 = \sqrt{c}k_0$.

Let $y = \sqrt{x}, g = \sqrt{c}, h = ga$. Then, $\ln\left(\frac{gy+1}{gy-1}\right) = ht + k_1$.

Let $K = e^{k_1}$. We get the simplified expression: $\frac{gy+1}{gy-1} = Ke^{ht}$ which can be transformed to: $gy+1=Ke^{ht}\left(gy-1\right)$ or $g\left(1-Ke^{ht}\right)y=-1-Ke^{ht}$ and $y=\dfrac{-1-Ke^{ht}}{g\left(1-Ke^{ht}\right)}$. $y=\sqrt{x}=\dfrac{Ke^{ht}+1}{g\left(Ke^{ht}-1\right)}$ which gives the desired equation $x(t)=\dfrac{\left(Ke^{ht}+1\right)^2}{c\left(Ke^{ht}-1\right)^2}$.

Let us determine K. We utilize the initial condition: $x_0=x(0)$. $\sqrt{x_0}=\dfrac{\left(Ke^0+1\right)}{\sqrt{c}\left(Ke^0-1\right)}$

which leads to $\sqrt{x_0}\sqrt{c}\left(K-1\right)=K+1$. Hence, $\left(\sqrt{x_0}\sqrt{c}-1\right)K=\sqrt{x_0}\sqrt{c}+1$ and finally

$K=\dfrac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}$. Now, we know how to determine K. Later, the sign and magnitude of K will be needed in the analysis. Do we know the sign of K?

$\left(\sqrt{x_0}>0\wedge\sqrt{c}>0\right)\Rightarrow\left(\sqrt{x}\sqrt{c}+1>0\right)$. Let us investigate the sign of $\sqrt{x_0}\sqrt{c}-1$. We assume that the value of x_0 makes sure that the increment is strictly positive. $\dfrac{dx}{dt}=a\sqrt{x}\left(1-cx\right)$. Then, we know that:

$$1-cx_0>0 \quad\Rightarrow\quad cx_0<1 \quad\Rightarrow\quad \sqrt{c}\sqrt{x_0}<\sqrt{1} \quad\Rightarrow\quad \sqrt{x_0}\sqrt{c}-1<0$$

As a result, we know that $K<0$. Do we know something about $|K|$?

$$K=\frac{\phi+1}{\phi-1} \quad 0<\phi=\sqrt{x_0}\sqrt{c}<1$$

.

$$K(\phi=\varepsilon)=\frac{\varepsilon+1}{\varepsilon-1} \quad\Rightarrow\quad \lim_{\varepsilon\to 0}K(\phi=\varepsilon)=-1.$$

$$\frac{dK}{d\phi}=\frac{(\phi-1)-(\phi+1)}{(\phi-1)^2} \quad\Rightarrow\quad \frac{dK}{d\phi}=\frac{-2}{(\phi-1)^2}<0$$

.

With this information, we know that $K < -1$. Now, we may determine $x(t)$ as an explicit function of x_0 and the parameters.

$$x(t) = \frac{\left(Ke^{ht}+1\right)^2}{c\left(Ke^{ht}-1\right)^2} \quad \wedge \quad K = \frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1} \quad \wedge \quad h = a\sqrt{c} \quad \Rightarrow$$

$$x(t) = \frac{\left(\left(\frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}+1\right)^2}{c\left(\left(\frac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}-1\right)^2}$$

Now, the dynamic properties of $x(t) = \frac{\left(Ke^{ht}+1\right)^2}{c\left(Ke^{ht}-1\right)^2}$ will be determined.

$$\frac{dx}{dt} = \left(\frac{2Khc}{c^2\left(Ke^{ht}-1\right)^4}\right)\left(\left(Ke^{ht}+1\right)\left(Ke^{ht}-1\right)^2 - \left(Ke^{ht}+1\right)^2\left(Ke^{ht}-1\right)\right)$$

$$\frac{dx}{dt} = \left(\frac{2Khc\left(Ke^{ht}+1\right)\left(Ke^{ht}-1\right)}{c^2\left(Ke^{ht}-1\right)^4}\right)\left(\left(Ke^{ht}-1\right) - \left(Ke^{ht}+1\right)\right)$$

$$\frac{dx}{dt} = \left(\frac{-4Khc\left(\left(Ke^{ht}\right)^2-1\right)}{c^2\left(Ke^{ht}-1\right)^4}\right)$$

We already know that $K < -1$. Hence, $\left(\left(Ke^{ht}\right)^2-1\right) > 0$. As a result, we find that $\frac{dx}{dt} > 0$.

$$\lim_{\substack{t\to\infty \\ h>0 \\ K<-1}} x(t) = \frac{\left(\dfrac{d\left(Ke^{ht}+1\right)^2}{dt}\right)}{\left(\dfrac{dc\left(Ke^{ht}-1\right)^2}{dt}\right)} = \frac{2\left(Ke^{ht}+1\right)hK}{2c\left(Ke^{ht}-1\right)hK} = \frac{\left(1+\dfrac{1}{Ke^{ht}}\right)}{c\left(1-\dfrac{1}{Ke^{ht}}\right)} = \frac{1}{c}$$

Hence, we know that, as $t \to \infty$, $x(t)$ monotonically converges to c^{-1}.

Results

A general dynamic function for the basal area of individual trees has been derived from a production theoretically motivated autonomous differential equation. The dynamic properties have been determined and monotone convergence has been proved. The differential equation is $\dfrac{dx}{dt} = a\sqrt{x}\left(1-cx\right)$, $a>0, c>0, 0<x<c^{-1}$ and the general dynamic function is: $x(t) = \dfrac{\left(\left(\dfrac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}+1\right)^2}{c\left(\left(\dfrac{\sqrt{x_0}\sqrt{c}+1}{\sqrt{x_0}\sqrt{c}-1}\right)e^{a\sqrt{c}t}-1\right)^2}$.

Discussion

The production theoretically motivated autonomous differential equation has recently been tested with regression analysis, using two sets of forest data from the Caspian forests and two sets of data from spruce forest in Sweden. Conference reports including these studies have been presented. They are described in more details below. The parameters obtained the expected signs and the t-values indicated very high precision. The adjusted R2-values, the very high F-values and the lack of unexplained trends in the residual graphs indicated that the functional form of the differential equation is relevant. The differential equation has been found to work very well with the empirical data from Iran and Sweden. Hopefully, it will be possible to test it also on forest data from other regions of the world. It is possible to use the model also under the influence of some forms of size dependent competition, via adjustments of the parameters (a,b). Further developements in this direction should be expected and some tests of alternative specifications have already been made.

Hatami et al. (2017) write:

“This study investigates basal area increment of individual trees in mixed species continuous cover forests in the north of Iran. Empirical data from the species *Fagus orientalis, Carpinus betulus, Parotia persica, Acer velutinum, Quercus castanifolia* and *Alnus subcordata* have been collected and analyzed with regression analysis. One general function for basal area increment has been estimated. Basal area increment is a statistically significant nonlinear function of the tree basal area and the competition.”

Mohammadi et al. (2017) write:

“The analysis shows that, for individual trees, there is a strongly significant nonlinear relation between the annual basal area increment, as the dependent variable, and the basal area of the individual trees. The results also show, that the basal area of competing trees, in the sample, has a positive influence on growth. The estimated result that increment is higher with more competing neighbor trees is probably obtained because of the following reason: The plots with higher volume per hectare, and more “competition”, most likely also have higher “site index” or “better soil” or “better forest production conditions” than the plot with lower volume per hectare.”

Lohmander et al. (2017) write:

“Economically optimal forest management decisions are determined within the continuous cover framework, in forests of Norway spruce in southern Sweden. The general methodological approach can be applied also to other species and in other regions. This study is based on a new growth function for individual trees of Norway spruce in southern Sweden managed with a continuous cover system. The function determines the diameter increment of individual trees based on the state of the tree itself, that is current diameter and crown status, and local competition in the neighborhood of each tree.”

Lohmander et al. (2017) use the differential equation developed in this paper as a part of a forest management optimization model and write:

"The expected present value of all harvests, over time and space, in a forest area, is maximized. Each tree is affected by competition from neighbor trees. The harvest decisions, for each tree, are functions of the prices in the stochastic market, the dimensions and qualities of the individual trees and the local competition. The adaptive control function, a diameter limit function, to be used in this forest is determined, that gives the maximum of the total expected present value of all activities over time. If the diameter of a particular tree is larger than the diameter limit, then the tree should be instantly harvested. Otherwise, it should be left for continued production. Three general forest management conclusions can be made: A tree should be harvested at a smaller diameter, if the local competition from other trees increases; a tree should be harvested at a larger diameter, if the predicted wood quality of the tree increases; a tree should be harvested at a smaller diameter, if the market net price for wood increases."

Conclusions

Differential equations can be very useful tools in forest management optimization. It is however important that they have theoretical and empirical support. Future efforts should be directed towards application of the suggested and derived function within forest management problems in several regions of the world.

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