Iranian Journal of Management Studies (IJMS) 2022, 15(1): 85-110
RESEARCH PAPER

A Lot-Sizing Model for Non-Instantaneous Deteriorating Products Under Advance Payment and Non-Linear Partial Backlogging

Sara Tavassoli[1], Neda Manavizadeh[2], Aida Rezaei[1], Masoud Rabbani[1*]

1. School of Industrial and Systems Engineering, College of Engineering, University of Tehran, Tehran, Iran
2. Department of Industrial Engineering, KHATAM University, Tehran, Iran

(Received: December 18, 2020; Revised: April 9, 2021; Accepted: May 16, 2021)

Abstract
In real life conditions, the buyers sometimes pay all or a percentage of the product price before receiving it, and the wholesaler sometimes allows them to prepay it at equal intervals. The present study develops a new mathematical model for products with non-instantaneous deteriorating rates by considering consecutive advance payments. In the proposed inventory model, the shortage is consisting of lost sales along with backorders simultaneously. In addition, the model considers the backlogging as totally dependent on the waiting time for the further cycle. In addition, the appropriate conditions to achieve the optimal solutions have been developed, and numerical instances have been provided to verify and evaluate the results and solution method. The useful methods to effectively reduce the annual total cost are provided according to the results of the sensitivity analysis.

Keywords: Economic order quantity, Non-linear partial backordering, Non-instantaneous deterioration, Advance payments, Deteriorating items

1. Introduction

A classic economic order quantity (EOQ) model is often known for its simplicity based on almost unrealistic assumptions that the usefulness of the product does not diminish during the replenishment cycle and product price has to be paid just after delivery. Generally, there are three different payment strategies, including (1) paying immediately after the products are received, (2) delayed payment, and (3) advance payment or prepayment. In practice, sometimes the purchasers pay all or a percentage of the price before the time of receipt, and the wholesaler sometimes allows them to prepay it at equal intervals**.** The prepayment of purchasing price has an important effect on the inventory control system, as the wholesaler cannot cancel the orders, and the buyers may get a price discount in return. Furthermore, the deterioration of most products in inventory does not occur as soon as their arrival in stock. In reality, most products have a time range during which the products are completely healthy and their quality does not change. After this time, the products will start to deteriorate based on a deterioration rate. This issue was first raised by Wu et al. (2006, p. 370), which is called "non-instantaneous deterioration." Therefore, controlling and maintaining the inventories of non-instantaneous deteriorating products is an important issue and should be considered when developing mathematical models.

[*] **Corresponding Author Email:** mrabani@ut.ac.ir

The literature of the present study includes three main parts such as deterioration, non-instantaneous deterioration, and advance payment, and each topic will be discussed in the following lines.

Storing deteriorating products is a vital issue that has been investigated by many researchers. Goyal and Giri (2001) reviewed studies on deteriorating products from the 1990s to 2001, and Bakker et al. (2012) reviewed this issue after 2001 based on the classification proposed by Goyal and Giri (2001). Nonetheless, the shortage has not been considered in the above models. Many pieces of research have addressed the concept of shortage for deteriorating products. Dye (2007) addressed a mathematical model for deteriorating products by the consideration of backordering and deteriorating rates. Hung (2011) extended Dye’s model to prove the main theorems in a more straightforward manner and validated his model for more demand function cases. Papachristos and Skouri (2000) developed a model with allowable shortage, which was partially backordered at a time-dependent rate. Skouri et al. (2011) presented an inventory formulation to handle deteriorating products considering late payments and partial backlogging. There are also numerous remarkable investigations on deterioration under shortage, such as Lee and Dye (2012), Wu et al. (2014), Taleizadeh and Nematollahi (2014), Viji and Karthikeyan (2018), Wang et al. (2014), etc. Moreover, some articles such as Bishi and Sahu (2018), Dutta and Kumar (2015), and Kumar and Keerthika (2018) investigated a mathematical model in which the partial backlogging was considered, and the deterioration rate was constant as well. A study conducted by Shah and Naik (2020) includes a new formulation for inventory problems where demand decreases as inventory shortage increases. Chakraborty et al. (2020) investigated the multi-product inventory model under storage capacity limitations, shortage, and inflation. Mishra et al. (2021) has investigated the effect of carbon emissions on sustainable inventory management. In this study, an inventory model has been presented to increase profits and considering the storage technology of deteriorating products and green technology to reduce carbon emissions. Halim et al. (2021) proposed an EPQ inventory model for deteriorating products to reduce the costs of the inventory system.

In the studies on deteriorating products, a significant assumption is that the deterioration of the products occurs just from the time they are delivered to the retailer. Musa and Sani (2012) proposed mathematical modeling for deteriorating products in which the products do not begin to deteriorate immediately, and they are stored under delayed payment situations. This topic has been extensively investigated. The presented model by Ghoreishi et al. (2015) for non-instantaneous deteriorating products includes partially backlogged, inflationary conditions, and delay in payments. Some papers such as Geetha and Udayakumar (2016), Mashud et al. (2019), and Soni and Suthar (2020) presented a new approach for storing non-instantaneous deteriorating products while concentrating on partially backlogging policy, and some other articles such as Ai et al. (2017), and Jaggi et al. (2017) studied the storage of non-instantaneous deteriorating products under completely backlogged shortages. In addition, Lashgari et al. (2018) and Mashud et al. (2020) investigated this issue by considering advanced and delayed payments under shortage conditions. Rezagholifam et al. (2020) investigated inventory and pricing under the conditions of the dependence of demand on inventory level and selling price for non-instantaneous deteriorating products. Maihami et al. (2021) evaluated the issue of pricing and inventory control for non-instantaneous deteriorating products under green investment conditions so that the expected profits to be maximized. To maximize the profit, Mahato and Mahata (2021) proposed an inventory model for non-instantaneous deteriorating products considering the expiration and deterioration time from qualitative and quantitative perspectives.

Although the strategies for paying the ordering cost with inventory problems were taken into account in the literature, the advance payment received considerable attention from researchers. Maiti et al. (2009) considered advance payment in a stochastic circumstance in which demand depends on the price. Gupta et al. (2009) solved a mathematical model for inventory problems with regard to a uniform rate of demand and a discount by paying an amount of money in advance. Taleizadeh et al. (2011) investigated a multi-product inventory problem by prepayment policy. Taleizadeh et al. (2013) addressed a mathematical formulation for inventory problem with sequential advance payments under the assumption that shortage is not acceptable, but backlog could be occurred both as partial and full. Taleizadeh (2014a) presented an inventory model under prepayment when the deterioration rate is constant. The shortage is permitted with complete backordering in this model. Taleizadeh (2014b) improved his previous work by considering partial backordering for evaporation products. Zhang et al. (2014) investigated an inventory-based problem by considering partial advanced payment and partial delayed payment at the same time. Zia and Taleizadeh (2015) studied the issue of inventory control by considering the advance and delayed payment and backlogging. Unlike the present study, they ignored the issue of non-instantaneous deteriorating and lost sales. Teng et al. (2016) considered a situation under advance payments where the deterioration rate increases simultaneously as the expiration date. Rajan and Uthayakumar (2017) and Sen et al. (2017) suggested an EOQ modeling by considering completely backlogged and credit payment. Sundararajan and Uthayakumar (2018) presented an EOQ approach for the products with a deteriorating rate regarding shortage and delay in payment policy simultaneously. Taleizadeh et al. (2018) presented a new formulation for backlogging inventory problems under the circumstances that full and partial prepayment, as well as partial delay in payment, are allowed. Khan et al. (2019) and Khan, Shaikh, Panda et al. (2020) addressed novel modeling for inventory problems under advance payment and partial backordering. Shaikh et al. (2019) addressed the dual-warehouse inventory model under trade credit policy and partial backlogging. By eliminating the impact of perishable items on each other through periodic inspections, Khakzad and Gholamian (2020) linked the deteriorating rate to the problem of inventory in the real world. In their study, the prepayment policy is provided by the supplier to the retailer, and unlike the present study, the issues of non-instantaneous deteriorating and partial backlogging are not considered. Liao et al. (2020) proposed inventory control by taking into account quality considerations and delays in the payment of purchase prices for deteriorating products. Udayakumar et al. (2020) evaluated the economic ordering policy under conditions of inflation, the dependence of demand on the purchasing price, and the delay in payment strategy for non-instantaneous deteriorating products. Das et al. (2021) studied the inventory model of deteriorating products considering the preservation technology and trade credit policy.

In the research of Diabat et al. (2017), the EOQ model has been optimized under the conditions of prepayment for the upstream level and delay in payment for the downstream level of the supply chain. In this study, shortage occurs as a backordering. Wu et al. (2018) examined the issue of inventory control of perishable products under the terms of the expiration date and advance-cash-credit payment policies. In the study by Khan, Shaikh, Konstantaras et al. (2020), two inventory models have been studied under the conditions of shortage, cost-dependent demand, and prepayment policy. Rahman et al. (2021) introduced the inventory model of perishable products under prepayment policy and maintenance technology. In this study, partial backlogging is allowed. The issues of inventory optimization, pricing under prepayment conditions, and price-dependent demand are presented in the study of Mashud et al. (2021). Shortage is not considered in this study. Unlike the present study, the issue of non-instantaneous deterioration has been ignored in these papers.

In the papers discussed in the first section of literature review, the assumptions of the classic EOQ model have been taken into account, and it has been assumed that product deterioration occurred as soon as entering the inventory system. The shortage has been considered in some mentioned papers, such as Bishi and Sahu (2018), Dutta and Kumar (2015), and Kumar and Keerthika (2018). The studies of the second part have adapted the assumption of the deterioration of products as soon as entering the inventory system to the real conditions and have studied the inventory policy for non-instantaneous deteriorating products. There is an assumption of shortage in some papers in this section, but the trade credit policy is not included. In the third part, some papers were reviewed that included the trade credit policy for deteriorating/non-instantaneous deteriorating products with and without considering shortage in the proposed inventory model. In the studies in this domain, the inventory for non-instantaneous deteriorating products has not been examined considering a combination of both shortages (backordering and lost sale) and advance payment policy simultaneously; this issue has been discussed in the present study.

A summary of relevant papers on deteriorating products under different types of payment strategies is presented in Table 1.

Table 1. Summarized Previous Literature

Reference	**Payment policy**		**Shortage**		**Deterioration**	
	Prepayment	**Delayed payment**	**Backordering**	**Partial backordering**	**Instantaneous**	**Non-instantaneous**
Ghoreishi et al. (2015)		✓		✓		✓
Teng et al. (2016)	✓				✓	
Geetha & Udayakumar (2016)				✓		✓
Sen et al. (2017)		✓	✓		✓	
Ai et al. (2017)			✓			✓
Jaggi et al. (2017)			✓			✓
Rajan & Uthayakumar (2017)		✓	✓		✓	
Kumar & Keerthika (2018)				✓	✓	
Sundararajan & Uthayakumar (2018)		✓	✓		✓	
Taleizadeh et al. (2018)	✓		✓		✓	
Khan et al. (2019)	✓			✓	✓	
Mashud et al. (2019)				✓		✓
Bishi & Sahu (2018)				✓	✓	
Soni & Suthar (2020)				✓		✓
Khan, Shaikh, Panda et al. (2020)	✓			✓	✓	
Shah & Naik (2020)			✓		✓	
Udayakumar et al. (2020)		✓				✓
Liao et al. (2020)		✓			✓	
Mahato & Mahata (2021)						✓
Das et al. (2021)		✓			✓	
Proposed model	✓			✓		✓

In the present paper, a mathematical model is studied in which some assumptions of the traditional EOQ are modified as follows: 1) deterioration does not start immediately after the retailer receives the products; hence, the formulation was developed for non-instantaneous deteriorating products, 2) a percentage of product cost must be paid in advance, 3) shortage is allowable and consists of lost sales along with backorders.

A mathematical model was proposed for real circumstances in which some assumptions of the traditional EOQ were modified.

- An inventory model was provided under advance payment for non-instantaneous deteriorating products.
- In the present study, partially backlogged shortages have been considered.
- It has been demonstrated that the cost in both decision variables is completely pseudo-convex.
- An optimal solution was presented for the mathematical model.

The rest of the present paper is structured as follows. The mathematical model for defined problems and its notation are developed in Section 2. In Section 3, some theorems and solution methods are established to achieve optimal replenishment policy and other research questions. In Sections 4 and 5, numerical instances and sensitivity analysis and their results are reported for proving the effectiveness of the solution method. Finally, the extracted managerial insights and conclusion are presented in Sections 6 and 7, respectively.

2. Problem Modeling

2.1. Notations and Assumptions

In order to present the formulation of the problem, some parameters and variables are adopted as follows.

	Parameters
A	Fixed ordering cost (per order)
C_p	Purchasing cost (per unit time)
C_h	Holding cost (per unit time)
C_b	Backlogging cost (per unit time)
G	Lost sale case (per unit time)
D	Constant demand rate in each period
θ	Constant deterioration rate
δ	Backordering parameter
n	Number of prepayments at equal intervals
α	Percentage of purchasing cost that should be paid in advance
t_d	The duration that the products do not face deterioration
L	Length of prepayments
i_1	Capital cost (per unit time)
Variables	
F	The fraction of demand that is filled from stock
T	Cycle time of replenishment
Q	Order quantity
B	Backordered quantity

Afterward, some assumptions are considered to establish the problem formulation.

1. The demand rate for products is constant and certain.
2. No deterioration occurs until t_d, and after this period, the products start to deteriorate at a constant rate of θ.
3. It is assumed that t_d is a given constant parameter. However, it is estimated by investigating the sample data of some random products in the past.
4. The horizon of planning is infinite.

5. As mentioned before, the shortage is allowed and includes lost sales along with backorders. In addition, the backordering rate is considered as totally dependent on the waiting time. In the following equation, $\beta(t)$ defines the backordering fraction such that t and δ are the waiting time up to the next cycle and the backordering, respectively: ($\beta(t) = 1/_{1+\delta t}$).
6. It is assumed that the buyer must prepay a fraction of the product cost and divide it into several equal-sized parts. The number of installed advance payments *n* is equally spaced and offered by the supplier.

2.2. Mathematical Model Formulation

An EOQ policy for non-instantaneous deteriorating products and consecutive prepayments with non-linear partial backordering is presented in this section. The inventory situation is described as follows. As shown in Figure 1, Q units of items arrive at the beginning of the period. The inventory amount will naturally decrease because of the demand rate over the period [0, t_d]. Afterward, it will drop to zero during the period $[t_d, FT]$ due to the demand and deterioration rate. After the inventory level meets zero at the end of the period, the retailer will face partial backlogging $[FT, T]$. The process of changing inventory levels is repeated in each replenishment cycle.

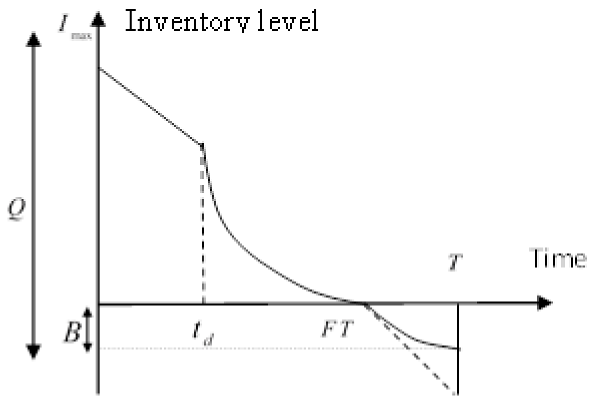

Figure 1. Diagram of Retailer Inventory Control

According to Figure 1, the changes in inventory levels over time are illustrated by the following differential equations:

$$\frac{dI_1(t)}{dt} = -D \quad ; \quad 0 \le t \le t_d \tag{1}$$

$$\frac{dI_2(t)}{dt} + \theta I(t) = -D \quad ; \quad t_d \le t \le FT \tag{2}$$

$$\frac{dI_3(t)}{dt} = -\frac{D}{1+\delta(T-t)} \quad ; \quad FT \le t \le T \tag{3}$$

Regarding the boundary conditions of $I(0) = I_{max}$ and $I(FT) = 0$ for the mentioned equations, the following equations are obtained:

$$I_1(t) = I_{max} - Dt \quad ; 0 \le t \le t_d \tag{4}$$

$$I_2(t) = \frac{D}{\theta}\left[e^{\theta(FT-t)} - 1\right] \quad ; t_d \le t \le FT \tag{5}$$

$$I_3(t) = -\frac{D}{\delta}\left[\ln\left[1+\delta\left(T-FT\right)\right] - \ln\left[1+\delta\left(T-t\right)\right]\right] \; ; FT < t \le T \tag{6}$$

As $I(t)$ has continuity at the time of t_d, the following equation is extracted from Equations 4 and 5:

$$I_1\left(t_d\right) = I_2\left(t_d\right) \rightarrow I_{\max} - Dt_d = \frac{D}{\theta}\left[e^{\theta(FT-t_d)} - 1\right] \tag{7}$$

Then the quantities are obtained as follows:

$$I_{\max} = Dt_d + \frac{D}{\theta}\left[e^{\theta(FT-t_d)} - 1\right] \tag{8}$$

$$B = -I_3\left(T\right) = \frac{D}{\delta}\ln\left[1+\delta\left(1-F\right)T\right] \tag{9}$$

$$Q = I_{\max} + B = Dt_d + \frac{D}{\theta}\left[e^{\theta(FT-t_d)} - 1\right] + \frac{D}{\delta}\ln\left[1+\delta\left(1-F\right)T\right] \tag{10}$$

The Taylor series expansion is applied for the exponential term ($e^{\theta(FT-t_d)} \approx 1+\theta\left(FT-t_d\right)+\theta^2(FT-t_d)^2/2$) since the deterioration rate value is inconsiderable in reality. After substituting into Equation 10 and simplifying it, the order level is obtained as follows:

$$Q = D\left[FT + \frac{\theta F^2T^2}{2} - \theta FTt_d + \frac{\theta t_d^{\,2}}{2} + \frac{\ln\left[1+\delta\left(1-F\right)T\right]}{\delta}\right] \tag{11}$$

Next, we derive various components of the cost function as the following sections show.

2.2.1. Purchasing Cost and Fixed Ordering Cost

As the order quantity is equal to $FT + \theta F^2T^2/2 - \theta FTt_d + \theta t_d^2/2 + ln\left[1+\delta\left(1-F\right)T\right]/\delta$, the cyclic purchasing cost is as follows:

$$CPC = C_pD\left\{FT + \frac{\theta F^2T^2}{2} - \theta FTt_d + \frac{\theta t_d^{\,2}}{2} + \frac{\ln\left[1+\delta\left(1-F\right)T\right]}{\delta}\right\} \tag{12}$$

2.2.2. Holding Cost

According to Figure 1, the cyclic holding cost is calculated by:

$$CHC = C_h \left\{ \int_0^{t_d} I_1(t)dt + \int_{t_d}^{FT} I_2(t)dt \right\} = C_h \left\{ \left. \frac{Dt^2}{2} \right|_0^{t_d} + \left. \frac{D}{\theta}\left(\frac{-e^{\theta(FT-t)}}{\theta} - t \right) \right|_{t_d}^{FT} \right\}$$

$$= C_h \left\{ \frac{Dt_d^{\,2}}{2} + \frac{D\left((1+\theta(FT-t_d) + \frac{1}{2}\theta^2(FT-t_d)^2) - \theta(FT-t_d) - 1 \right)}{\theta^2} \right\} \tag{13}$$

$$= C_h D\left(t_d^{\,2} - FTt_d + \frac{1}{2}F^2T^2 \right)$$

2.2.3. Total Shortage Cost

As depicted in Figure 1, the vendor faces two kinds of shortages over the period: lost sales and backorders. The cyclic inventory shortage cost due to the backorder and lost sales are calculated by:

$$CBC = C_b \int_{FT}^{T} I_3(t)dt = \frac{C_b D}{\delta^2}\left\{ (1-F)T\delta - \ln\left[1+\delta(1-F)T \right] \right\} \tag{14}$$

$$CLC = G\int_{FT}^{T} D\left[1-\beta(T-t) \right] dt$$
$$= \frac{GD}{\delta}\left\{ (1-F)T\delta - \ln\left[1+\delta(1-F)T \right] \right\} \tag{15}$$

2.2.4. Capital Cost

The retailer gets a loan from a bank to pay α percent of the product cost as prepayment in *n* equal installments before delivering the order and $1-\alpha$ percent of the product cost as the time of delivery. Since the grey area in Figure 2 is the combination of *n* rectangles, the summation of all areas yields the buyer's cyclic capital cost.

$$CCC = \left(i_1 \frac{\alpha C_p Q}{n} \times n \times \frac{L}{n} \right) + \left(i_1 \frac{\alpha C_p Q}{n} \times (n-1) \times \frac{L}{n} \right)$$
$$+ \ldots + \left(i_1 \frac{\alpha C_p Q}{n} \times (n-(n-1)) \times \frac{L}{n} \right)$$
$$= \left(i_1 \frac{\alpha C_p Q}{n}\frac{L}{n} \right)[n+(n-1)+\ldots\ldots+2+1]$$
$$= \left(i_1 \frac{\alpha C_p D\left[FT + \frac{\theta F^2T^2}{2} - \theta FTt_d + \frac{\theta t_d^{\,2}}{2} + \frac{D}{\delta}\ln\left[1+\delta(1-F)T \right] \right]}{n}\frac{L}{n} \right)\frac{n(n+1)}{2} \tag{16}$$
$$= i_1 \alpha C_p D\left[FT + \frac{\theta F^2T^2}{2} - \theta FTt_d + \frac{\theta t_d^{\,2}}{2} + \frac{\ln\left[1+\delta(1-F)T \right]}{\delta} \right]\frac{(n+1)}{2n}L$$

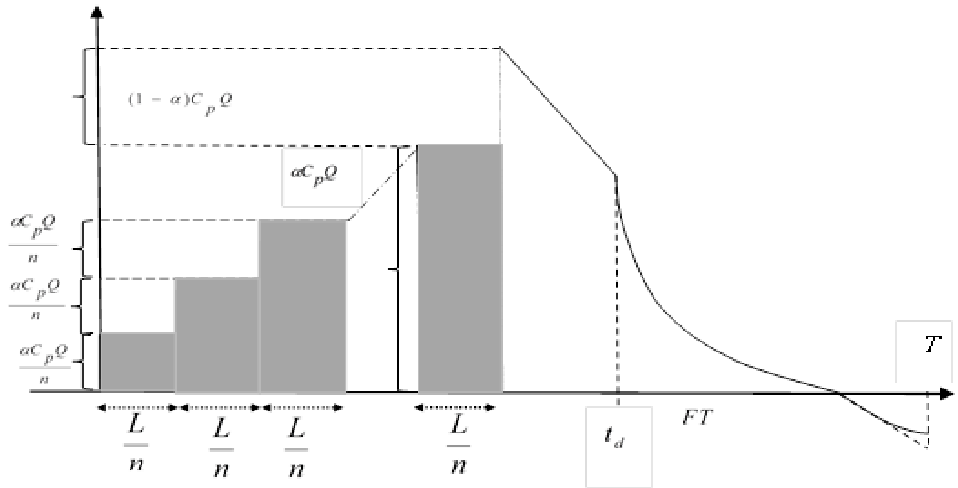

Figure 2. Paying the Retailer Interest to the Bank for Payment in Advance

2.2.5. Objective Function

Therefore, according to the components of objective function calculated previously, the retailer's entire inventory cost per period and year are respectively equal to:

$$CTC=\left(\begin{array}{l}\underbrace{A}_{\text{fixed cost}}+\underbrace{i_1\alpha C_pD\left[\begin{array}{l}FT+\dfrac{\theta F^2T^2}{2}-\theta FTt_d\\+\dfrac{\theta t_d^{\,2}}{2}+\dfrac{1}{\delta}\ln\left[1+\delta(1-F)T\right]\end{array}\right]\dfrac{(n+1)}{2n}L}_{\text{capital cost}}\\ \underbrace{+C_pD\left[\begin{array}{l}FT+\dfrac{\theta F^2T^2}{2}-\theta FTt_d+\dfrac{\theta t_d^{\,2}}{2}\\+\dfrac{1}{\delta}\ln\left[1+\delta(1-F)T\right]\end{array}\right]}_{\text{purchasing cost}}\underbrace{+C_hD\left(\begin{array}{l}t_d^{\,2}-FTt_d\\+\dfrac{1}{2}F^2T^2\end{array}\right)}_{\text{holding cost}}\\ \underbrace{+\dfrac{C_bD}{\delta}\left(\begin{array}{l}(1-F)T\\-\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta}\end{array}\right)+GD\left(\begin{array}{l}(1-F)T\\-\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta}\end{array}\right)}_{\text{shortage cost}}\end{array}\right) \tag{17}$$

$$ATC=\left(\begin{array}{l}\underbrace{\dfrac{A}{T}}_{\text{fixed cost}}+\underbrace{i_1\alpha C_pD\left[\begin{array}{l}F+\dfrac{\theta F^2T}{2}-\theta Ft_d\\+\dfrac{\theta t_d^{\,2}}{2T}+\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta T}\end{array}\right]\dfrac{(n+1)}{2n}L}_{\text{capital cost}}\\ \underbrace{+C_pD\left[\begin{array}{l}F+\dfrac{\theta F^2T}{2}-\theta Ft_d+\dfrac{\theta t_d^{\,2}}{2T}\\+\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta T}\end{array}\right]}_{\text{purchasing cost}}\underbrace{+C_hD\left(\begin{array}{l}\dfrac{t_d^{\,2}}{T}-Ft_d\\+\dfrac{1}{2}F^2T\end{array}\right)}_{\text{holding cost}}\\ \underbrace{+\dfrac{C_bD}{\delta}\left(\begin{array}{l}(1-F)\\-\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta T}\end{array}\right)+GD\left(\begin{array}{l}(1-F)\\-\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta T}\end{array}\right)}_{\text{shortage cost}}\end{array}\right) \tag{18}$$

3. Theoretical Results and Optimal Solutions

According to Cambini and Martein (2008), the fraction $q(x) = f(x)/g(x)$ is totally pseudo-convex where $f(x)$ is determined as a non-negative differentiable function and completely convex as well, and $g(x)$ is determined as a positive, differentiable, and concave function. Hence the optimum solution (T^*, F^*) can be found as the whole cost is minimized using aforementioned theoretical results. In fact, by applying the above theoretical result, we show that the annual total cost function is a strictly pseudo-convex function of variables F and T, and simplifies the unique optimal solution to a local minimum.

So let us first define L_1 as follows:

$$L_1 = \left(i_1 \alpha C_p D \frac{(n+1)}{2n} L + C_p D \right) \left(\begin{array}{l} \frac{\theta F^2}{2} - \frac{\ln\left[1+\delta(1-F)t_d\right]}{\delta t_d{}^2} \\ -\frac{\theta}{2} + \frac{(1-F)}{t_d\left(1+\delta(1-F)t_d\right)} \end{array} \right)$$
$$+ C_h D \left(\frac{1}{2}F^2 - 1\right) + \left(\frac{C_b D}{\delta} + GD\right) \left(\begin{array}{l} \frac{\ln\left[1+\delta(1-F)t_d\right]}{\delta t_d{}^2} \\ -\frac{(1-F)}{t_d\left(1+\delta(1-F)t_d\right)} \end{array} \right) - \frac{A}{t_d{}^2} \tag{19}$$

Thus, we have the following results:

Theorem 1. For any given F if $\frac{C_b}{\delta} + G - \left(i_1 \alpha C_p \frac{(n+1)}{2n} L + C_p \right) \geq 0$:

(I) $ATC\ (F,T)$ is an entirely pseudo-convex in T; thus, T^* refers to a minimum unique solution.

(II) If $L_1 \geq 0$, there is a unique $T \in [t_d, \infty]$, such that $ATC\ (F,T)$ can be minimized.

(III) If $L_1 < 0$, then $ATC(F,T)$ would be minimized at $T = t_d$ during the interval.

Proof. Refer to Appendix A.

In order to obtain T^*, the first partial derivative of $ATC\ (F,T)$ with respect to T is taken and put into zero, and then the terms are re-arranged; hence:

$$-\frac{A}{D} + \left[C_h + \theta C_p \left(i_1 \alpha \frac{(n+1)}{2n} L + 1 \right) \right] \times \left(\frac{(FT)^2 - t_d^2}{2} \right)$$
$$+ \left[C_p \left(i_1 \alpha \frac{(n+1)}{2n} L + 1 \right) - \left(\frac{C_b}{\delta} + G \right) \right]$$
$$\times \left[\frac{\ln\left[1+\delta(1-F)T\right]}{\delta} + \frac{(1-F)T}{\left(1+\delta(1-F)T\right)} \right] = 0 \tag{20}$$

Similarly, in this section we define the values of L_2 and L_3 as:

$$L_2 = -D \left[\begin{array}{l} C_p \left[t_d \theta - \left(\frac{\delta T}{1+\delta T} \right) \right] \left(i_1 \alpha \frac{(n+1)}{2n} L + 1 \right) + t_d C_h \\ + \left(\frac{C_b + G\delta}{\delta} \right) \left(\frac{\delta T}{1+\delta T} \right) \end{array} \right] \tag{21}$$

$$L_3 = D\left(T - t_d\right)\left[\theta C_p\left(i_1\alpha\frac{(n+1)}{2n}L + 1\right) + C_h\right] \tag{22}$$

We know from Appendix B that $L_2 < L_3$, so we can mathematically prove the following results:

Theorem 2. For any value of T :

(I) $ATC\ (F,T)$ is entirely convex in F; thus, there is a unique F^*.

(II) If $L_2 > 0$ and $L_3 \geq 0$, so $ATC\ (F,T)$ will be minimized at F^*=0.

(III) If $L_2 > 0,\ L_3 < 0$, so there is a unique $F^* \in (0,1)$ in which $ATC\ (F,T)$ will be minimized.

(IV) If $L_2 < 0$ and $L_3 \leq 0$, so $ATC\ (F,T)$ will be minimized at F^*=1.

Proof. Refer to Appendix B.

For any value of T, the first partial derivative of $ATC\ (F,T)$ with respect to F is taken and put into zero, and the terms are re-arranged; hence:

$$\begin{bmatrix} \left(\theta C_p\left(i_1\alpha\frac{(n+1)}{2n}L+1\right)+C_h\right)(FT-t_d) \\ +\left[C_p\left(i_1\alpha\frac{(n+1)}{2n}L+1\right)-\frac{C_b}{\delta}-G\right]\left(\frac{\delta(1-F)T}{1+\delta(1-F)T}\right) \end{bmatrix} = 0 \tag{23}$$

Next, we suggested an algorithm to achieve the optimal solution (F, T) by considering Theorem 1 and 2.

3.1. An Algorithm to Achieve the Optimal Solutions

An algorithm to simultaneously optimize F and T is presented as follows:

(Step 1) Solve Equations (20) and (23) using input data and compute T and F simultaneously.

(Step 2) If $T \geq t_d$ and $0 \leq F \leq 1$, put $T^* = T, F^* = F$ and stop the procedure.

(Step 3) If $T \geq t_d$ and $F < 0$, put $T^* = T, F^* = 0$ and stop the procedure.

(Step 4) If $T \geq t_d$ and $F > 1$, put $T^* = T, F^* = 1$ and stop the procedure.

(Step 5) If $T < t_d$ and $0 \leq F \leq 1$, put the optimal solution $T^* = t_d$, calculate L_2 and L_3, find F^* based on the outcomes of Theorem 2, and stop the procedure.

(Step 6) If $T < t_d$ and $F < 0$, put $T^* = t_d, F^* = 0$ and stop the procedure.

(Step 7) If $T < t_d$ and $F > 1$, put $T^* = t_d, F^* = 1$ and stop the procedure.

As mentioned earlier, the total annual cost in both of the two decision variables T and F is strictly pseudo-convex; thus, the optimal values of these two variables are calculated using the steps of the proposed algorithm while ensuring the lowest annual cost. According to the abovementioned algorithm, first, the values of T and F were calculated using Equations (20) and (23), and then their optimal values were determined according to the algorithm steps. After obtaining the T^*, F^* and ATC^* is obtained by Equation (18). Finally, with regard to $T^*$and F^*, the optimal value of Q is calculated by Equation (11). Figure 3 indicates the flowchart of the algorithm.

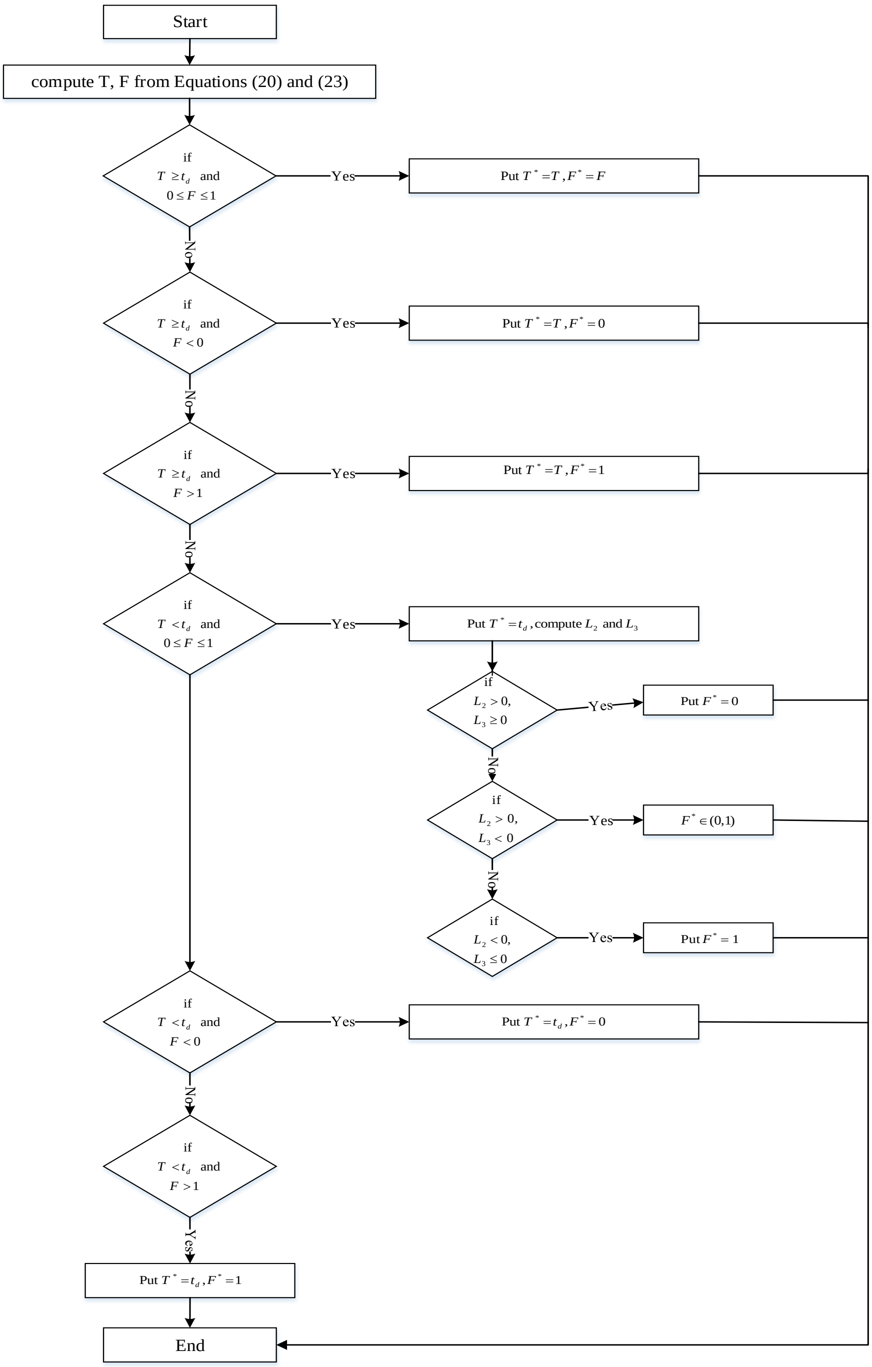

Figure 3. The Flowchart of Algorithm

4. Numerical Examples

In this section, several numerical instances were applied using MATHEMATICA 10.2 to evaluate and verify the presented model and solution method. Mathematica software has created a simple and easy way to program mathematical functions and equations compared to MATLAB. This software can be used in various areas and has an extensive scope of application. Mathematics has created effective programming for computation, and its learning curve is more straightforward than MATLAB (Educba, 2020). Considering a set of features of both software packages, Mathematics has provided a suitable platform for solving the proposed model; therefore, this software has been applied in the present research.

Example 1. Let D=250 unit/year, A=\$250, C_h=\$10 per unit, C_p=\$20 per unit, G=5, C_b=\$10 per unit, n=5, $i_1 = 0.13$/\$/year, $\delta = 0.2$, $\theta = 0.3$, L= 0.08 years, α= 0.5, and t_d=0.08 years.

The optimal cycle time is $T^* = 0.588$ years, and the value of the fraction of demand that is filled with stock is $F^* = 0.386$. As both decision variables are optimized between their feasible lower and upper bound, they are accepted as unique global solutions for the problem. Therefore, the optimal order quantity for each cycle is equal to $Q^* = 144.584$ units, and the minimum annual cost per time unit is $ATC^* = \$5695.122$.

Example 2. D=400 unit/year, A=\$500 order, C_h=\$12 per unit, C_p=\$10 per unit, G=4, C_b=\$12 per unit, n=4, $i_1 = 0.15$/\$/year, $\delta = 0.8$, $\theta = 0.18$, L= 0.25 years, α= 0.4, and t_d=0.1 years.

The results of example 2 are: $T^* = 0.397$, $F^* = 0.412$, $Q^* = 151.105$, and $ATC^* = \$5566.6842$.

Example 3. D=800 unit/year, A=\$300 order, C_h=\$4 per unit, C_p=\$15 per unit, G=8, C_b=\$15 per unit, n=6, $i_1 = 0.25$/\$/year, $\delta = 0.35$, $\theta = 0.08$, L= 0.3 years, α= 0.6, and t_d=0.25 years.

The results of example 3 are: $T^* = 0.238$, $F^* = 1$, $Q^* = 190.404$, and $ATC^* = \$13996.87$.

Example 4. D=1000 unit/year, A=\$100order, C_h=\$10 per unit, C_p=\$30 per unit, G=5, C_b=\$20 per unit, n=3, $i_1 = 0.10$/\$/year, $\delta = 0.45$, $\theta = 0.4$, L= 0.17 years, α= 0.4, and t_d=0.15 years.

The results of example 4 are: $T^* = 0.1602$, $F^* = 0.2131$, $Q^* = 228.085$, and $ATC^* = \$32805.947$.

5. Sensitivity Analysis

In this section, a comprehensive sensitivity analysis is provided to investigate the effect of changes of parameters on the model's behavior. Thus, the analysis is accomplished by considering the data of Example 1 and changing some parameters. Table 2 indicates the outcomes of T^*, F^*, Q^*, and ATC^* by decreasing or increasing each of the relevant and important parameters in the system t_d, θ, α and δ by -50%, -25%, +25%, and +50%.

Table 2. Results of Sensitivity Analysis

Parameters	T^*	F^*	Q^*	ATC^*
t^d=0.04	0.598	0.335	146.658	5742.499
t^d=0.06	0.592	0.361	145.464	5726.468
t^d=0.110	0.584	0.411	144.011	5701.553
t^d=0.04	0.583	0.437	143.738	5692.346
$\theta = 0.15$	0.585	0.426	143.972	5681.301
$\theta = 0.225$	0.586	0.404	144.359	5688.900
$\theta = 0.375$	0.588	0.370	144.696	5701.140
$\theta = 0.45$	0.589	0.356	144.729	5706.240
$\alpha = 0.25$	0.620	0.391	152.449	5688.953
$\alpha = 0.375$	0.622	0.371	152.552	5693.906
$\alpha = 0.625$	0.623	0.339	152.898	5706.794
$\alpha = 0.75$	0.624	0.326	153.203	5713.754
$\delta = 0$	0.500	0.478	125.940	5802.859
$\delta = 0.1$	0.535	0.439	133.457	5762.874
$\delta = 0.3$	0.669	0.323	160.967	5649.485
$\delta = 0.4$	0.813	0.248	187.715	5566.294

According to Table 2, the decision variable F^* is more sensitive to changes in parameter values of t_d, θ, α, and δ compared to other variables. However, ATC^* and decision variables T^* and Q^* are less sensitive to changes in parameter values of t_d , θ, and α , and they are more sensitive to changes of δ . As shown in Figure 4, increasing the value of the parameter t_d leads to the decrease in T^* and Q^* but an increase in the variable F^*. Moreover, according to Figure 5, (ATC^*) reduces when the length of time that the products do not face deterioration (t_d) increases; this illustrates that if a traditional EOQ comes close to reality and considers non-instantaneous deteriorating products, it will be able to reduce inventory costs. If the buyer can increase t_d , the cost of inventory will be significantly reduced.

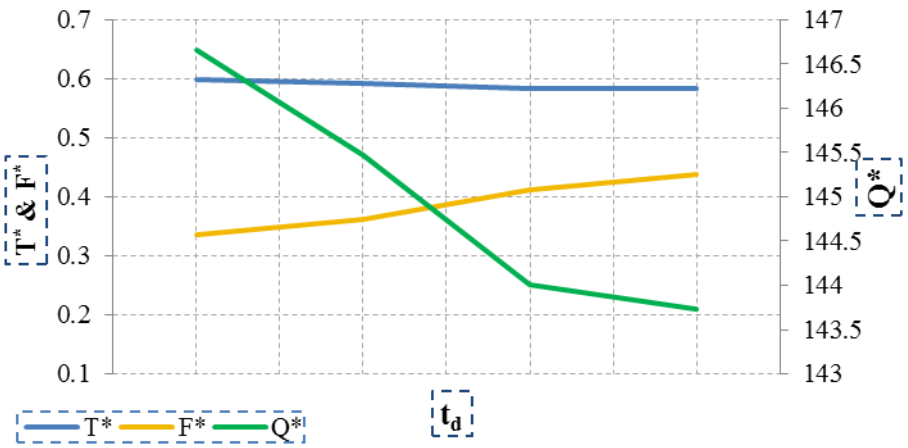

Figure 4. Behaviors of Decision Variables Based on Changing the Parameter t_d

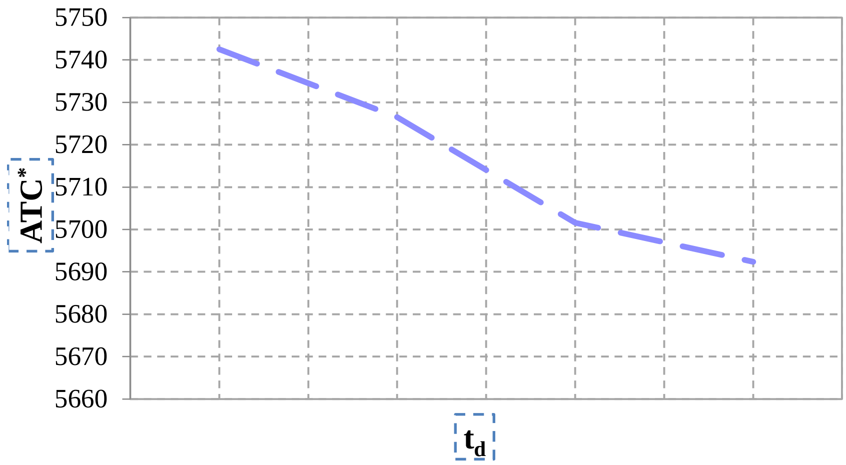

Figure 5. Sensitivity Analysis of Annual Total Cost Based on Changing the Parameter t_d

According to Figures 6 and 7, increasing the value of the parameter θ leads to a decrease in the value of F^*, but T^*, Q^*, and ATC^*increase; this indicates that improving storage conditions and reducing deterioration rates will lead to improved and reduced costs.

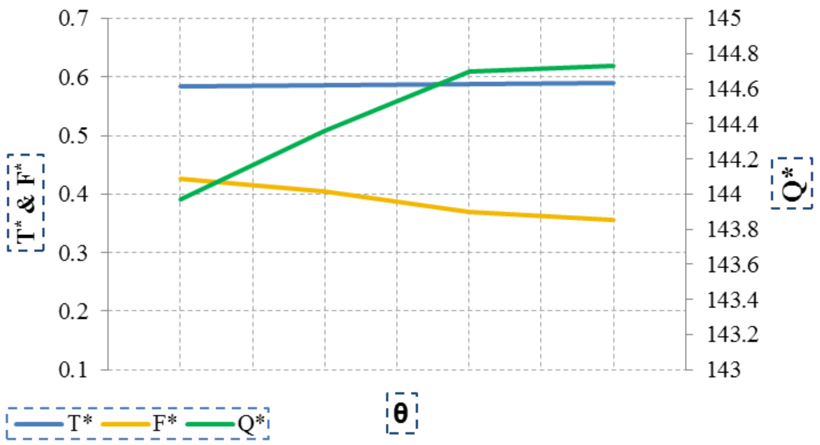

Figure 6. Behaviors of Decision Variables Based on Changing the Parameter θ

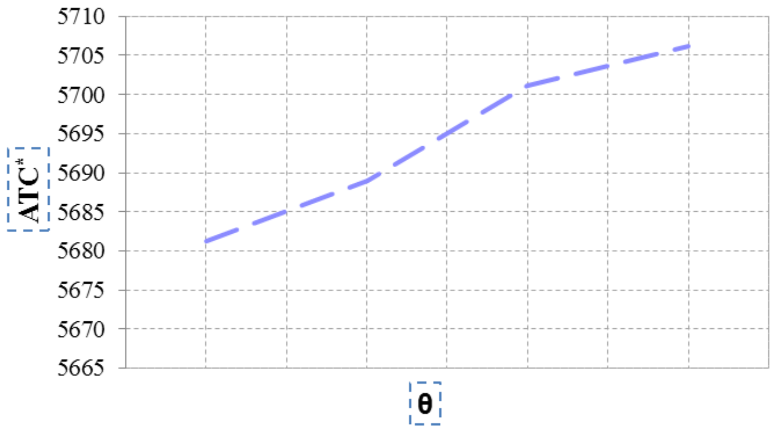

Figure 7. Sensitivity Analysis of Annual Total Cost Based on Changing the Parameter θ

When the value of α increases, T^*, Q^*, and ATC^* increase while F^* experiences a reduction (see Figures 8 and 9). This specifies that increasing the prepaid purchasing cost has a strong impact on the annual total cost; hence, the decision-maker should attempt to select a supplier who demands a lower percentage of prepayment. In this way, inventory costs are improved by reducing the prepayment costs.

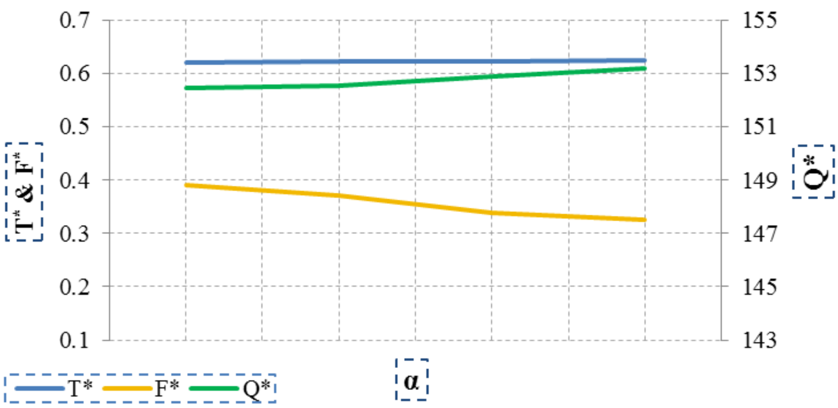

Figure 8. Behaviors of Decision Variables Based on Changing the Parameter α

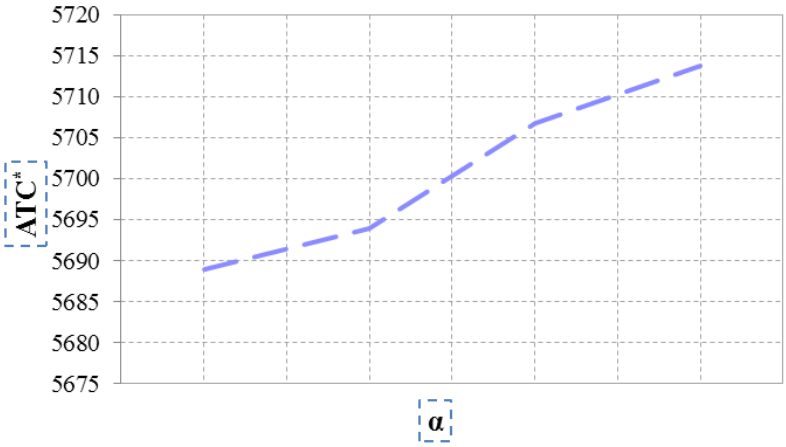

Figure 9. Sensitivity Analysis of Annual Total Cost Based on Changing the Parameter α

Figures 10 and 11 indicate that the decision variables are highly sensitive to changes in δ. Increasing δ leads to an increase in T^* and Q^* but a decrease in F^* and ATC^*. Significant changes in decision variables and objective function indicate that improving the shortage rate (by increasing δ) will lead to improved inventory costs.

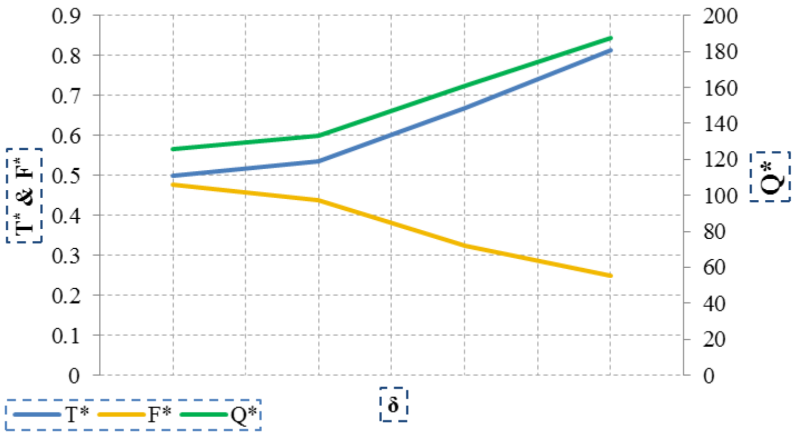

Figure 10. Behaviors of Decision Variables Based on Changing the Parameter δ

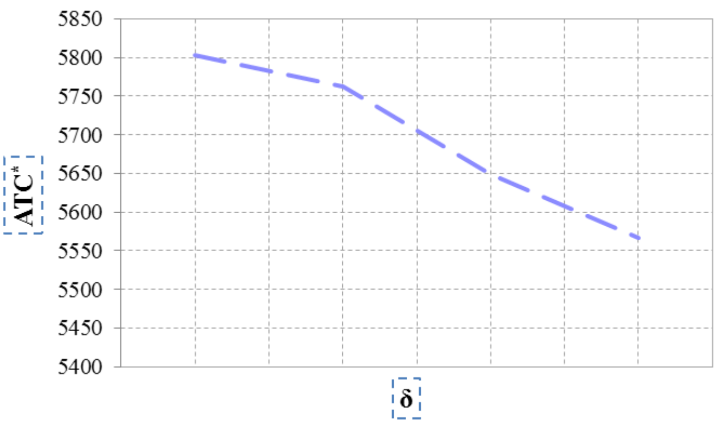

Figure 11. Sensitivity Analysis of Annual Total Cost Based on Changing the Parameter δ

In general, the results presented in Table 2 and the foregoing figures reveal that:

- T^*, Q^*, and ATC^* decrease and F^* increases with the rise of t_d. In other words, the later the deterioration occurs, the lower the annual total cost will be (See figures 4 and 5).
- By increasing the parameter θ, the value of F^* decreases, but T^*, Q^*, and ATC^* increase. In other words, the annual total cost has a higher value in the large values of deterioration rate (See figures 6 and 7).
- T^*, Q^*, and ATC^* increase and F^* decreases with an increase in the value of α (See figures 8 and 9). Besides, T^* and Q^* are inconsiderably sensitive to the values of the parameters α and θ.
- Increasing the parameter δ leads to an increase in T^* and Q^* and a decrease in F^* and ATC^*. Moreover, T^*, F^*, Q^*, and ATC^* are significantly sensitive to changes in δ; hence, the backordering rate plays an important role in minimizing the total annual profit (See figures 10 and 11).
- A higher value of t_d and the lower values of θ and α usually lead to the lower value of ATC^*, T^*, and Q^*, whereas a higher values of F^* and δ cause higher values of T^* and Q^* and lower values of ATC^* and F^*.

6. Main Managerial Insights

According to the findings of the sensitivity analysis, some managerial insights are presented as follows.

- As shown in Figure 5, annual costs are reduced by increasing the parameter t_d, which reveals that in the real world, by improving their storage conditions, the deterioration of products can be postponed; in fact t_d increases. Therefore, managers need to make a comprehensive analysis of the profit and costs of postponement to make a proper decision to improve storage conditions.
- According to Figure 7, annual costs increase as the deterioration rate increases; hence, managers should pay attention to the use of procedures and policies to reduce the deterioration rate and costs consequently.
- As indicated in Figure 9, the purchaser’s annual cost increases as the fraction of the product price to be prepaid increases. Thus, due to the sensitivity of the mathematical model to the parameter α, the buyer should attempt to reduce it. From the managerial perspective, the buyer should try to buy the items from a supplier who demands a lower prepayment percentage. In this case, the buyer takes lower-interest-rate bank loans and, as a result, incurs lower capital costs.
- According to Figure 11, the annual cost decreases when the parameter δ increases. Therefore, managers need to improve the shortage rate to maximize profits and reduce costs.
- The best case for the buyer happens under the conditions of the lowest deterioration rate, the highest time that the products do not face deterioration, the lowest amount to pay in advance, and the improvement of the shortage rate. Therefore, managers should pay attention to the latest techniques and policies to decrease the deterioration rate and increase the time that the products do not experience deterioration. Moreover, they

should negotiate with suppliers to improve prepayment policy and reduce the fraction of product price to be paid in advance.

As a result of managerial insights, it can be said that the present study provided a suitable approach for managers to solve their inventory decisions in real conditions. Applying the proposed framework, the inventory model can be investigated under prepayment and shortage conditions, and efficient decisions can be made to control issues related to non-instantaneously deteriorating products. The present research reveals that the joint implementation of prepayment policy and non-instantaneously deteriorating products allows better decision-making to properly manage and control the inventory flow.

In fact, it is necessary to form an appropriate inventory policy considering various factors from an optimal point of view because inventory management may affect the entire supply chain. Holding large amounts of inventory for long periods is usually not profitable because it imposes maintenance and deterioration costs on the organization. On the other hand, holding few amounts of inventory is not profitable since it would cause frequent shortages at the time of high demand and increase the risk of losing sales. The shortage is another effective factor in managing an optimal inventory control policy. When shortages occur, some customers tend to wait, and some others turn to different options. Thus, considering a combination of types of model shortages brings it closer to the real condition. Moreover, it is true that most products deteriorate throughout storage and their original value is reduced or lost, but in the real world, products retain their value for a while and do not deteriorate. However, most previous researchers have assumed that deterioration of products occurs as soon as they enter the inventory system. Besides, a prepayment policy is provided by the buyer for many seasonal fruits and vegetables; the buyers pay a part of the purchase price to receive the order from suppliers on time, thus reduce the risk of order cancellation. Therefore, the consideration of real assumptions, including deterioration, shortage, and trade credit policies such as prepayment, plays an important role in an appropriate inventory control system. Therefore, the nature of these characteristics is an important part of inventory modeling, which is included in the present model. The proposed model will assist managers with properly incorporating real-world characteristics and assumptions into the classical inventory control policy and managing inventory under appropriate conditions.

It is noteworthy that in real-life conditions for the seasons when the demand for fruits and vegetables is higher, the buyer should purchase more products and store them in temporary warehouses with appropriate technology; this issue can be considered in real conditions but has not been included in the proposed model. Moreover, certain circumstances were assumed in the proposed model, but the parameters such as demand and deterioration rate are mainly uncertain and variable in real conditions. These limitations can lead to appropriate suggestions for developing the present model.

7. Conclusion

It is observed in many real-world inventory models that deteriorating products do not begin to deteriorate as soon as entering the inventory system, and this deterioration occurs under different conditions and after a certain period of time. Nonetheless, this issue has been ignored in most classic EOQ models. Moreover, some considerations such as the shortage and prepayment of purchase prices by buyers are among the issues that effectively influence inventory decisions in real conditions. Therefore, a modified EOQ model was proposed for non-instantaneously deteriorating products to minimize annual total costs under advance payment policy and non-linear partial backlogging. In this model, it was assumed that the products are of non-instantaneously deteriorating type; they do not deteriorate within a period of time from their entry into the inventory system and then decline at a constant rate. The trade credit policy is also provided to buyers as a prepayment of the purchase price, and the inventory shortage has been presented as a combination of the backordering and lost sale shortages. It was demonstrated that the total cost in both decision variables is entirely pseudo-convex. Consistent with the theoretical findings, the necessary and sufficient qualifications could be provided to achieve the optimal value. Finally, variant numerical instances were applied to show the effectiveness of this approach, and the sensitivity analysis was investigated to provide some main managerial insights. According to the results of the model, it can be concluded that the ideal situation to enhance inventory costs is to provide appropriate conditions to reduce the deterioration rate, increase the delay time for deterioration, and reduce the prepayment percentage of the purchase price. The proposed model creates an effective insight by taking into account the common assumptions in the real world for making desirable decisions in managing the inventory of non-instantaneously deteriorating products.

The authors suggest that, in future works, researchers might extend the proposed model in the present study by incorporating it into other assumptions. It is hoped that the applied assumptions be more realistic and practicable. For example, the supplier may offer an authorized delay in payment for the remaining unpaid purchasing cost to use a hybrid payment strategy. Another future direction can consider the number of prepayments before delivery and the length of time while prepayments are paid as other decision variables. Finally, special sales, discount schemes, or pricing policies might be incorporated into the model.

References

Ai, X. Y., Zhang, J. L., & Wang, L. (2017). Optimal joint replenishment policy for multiple non-instantaneous deteriorating items. *International Journal of Production Research*, *55*(16), 4625-4642. https://doi.org/10.1080/00207543.2016.1276306

Bakker, M., Riezebos, J., & Teunter, R. H. (2012). Review of inventory systems with deterioration since 2001. *European Journal of Operational Research*, *221*(2), 275-284. https://doi.org/10.1016/j.ejor.2012.03.004

Bishi, B., & Sahu, S. K. (2018). An inventory model for deteriorating items with quadratic demand and partial backlogging. *Journal of Computer and Mathematical Sciences*, *9*(12), 2188-2198.

Cambini, A., & Martein, L. (2008). *Generalized convexity and optimization: Theory and applications*. Springer Science & Business Media.

Chakraborty, D., Jana, D. K., & Roy, T. K. (2020). Multi-warehouse partial backlogging inventory system with inflation for non-instantaneous deteriorating multi-item under imprecise environment. *Soft Computing*, *24*(19), 14471-14490.

Das, S. C., Manna, A. K., Rahman, M. S., Shaikh, A. A., & Bhunia, A. K. (2021). An inventory model for non-instantaneous deteriorating items with preservation technology and multiple credit periods-based trade credit financing via particle swarm optimization. *Soft Computing*, …, 5365-5384.

Diabat, A., Taleizadeh, A. A., & Lashgari, M. (2017). A lot sizing model with partial downstream delayed payment, partial upstream advance payment, and partial backordering for deteriorating items. *Journal of Manufacturing Systems*, *45*, 322-342. https://doi.org/10.1016/j.jmsy.2017.04.005

Dutta, D., & Kumar, P. (2015). A partial backlogging inventory model for deteriorating items with time-varying demand and holding cost. *International Journal of Mathematics in Operational Research*, *7*(3), 281-296. https://doi.org/10.1504/IJMOR.2015.069144

Dye, C. Y. (2007). Joint pricing and ordering policy for a deteriorating inventory with partial backlogging. *Omega*, *35*(2), 184-189. https://doi.org/10.1016/j.omega.2005.05.002

Educba. (2020). Mathematica vs Matlab. Accessed from https://www.educba.com/mathematica-vs-matlab/

Geetha, K. V., & Udayakumar, R. (2016). Optimal lot sizing policy for non-instantaneous deteriorating items with price and advertisement dependent demand under partial backlogging. *International Journal of Applied and Computational Mathematics*, *2*(2), 171-193.

Ghoreishi, M., Weber, G. W., & Mirzazadeh, A. (2015). An inventory model for non-instantaneous deteriorating items with partial backlogging, permissible delay in payments, inflation-and selling price-dependent demand and customer returns. *Annals of Operations Research*, *226*(1), 221-238.

Goyal, S. K., & Giri, B. C. (2001). Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, *134*(1), 1-16. https://doi.org/10.1016/S0377-2217(00)00248-4

Gupta, R. K., Bhunia, A. K., & Goyal, S. K. (2009). An application of Genetic Algorithm in solving an inventory model with advance payment and interval valued inventory costs. *Mathematical and Computer Modelling*, *49*(5-6), 893-905. https://doi.org/10.1016/j.mcm.2008.09.015

Halim, M. A., Paul, A., Mahmoud, M., Alshahrani, B., Alazzawi, A. Y., & Ismail, G. M. (2021). An overtime production inventory model for deteriorating items with nonlinear price and stock dependent demand. *Alexandria Engineering Journal*, *60*(3), 2779-2786. https://doi.org/10.1016/j.aej.2021.01.019

Hung, K. C. (2011). An inventory model with generalized type demand, deterioration and backorder rates. *European Journal of Operational Research*, *208*(3), 239-242. https://doi.org/10.1016/j.ejor.2010.08.026

Jaggi, C. K., Tiwari, S., & Goel, S. K. (2017). Credit financing in economic ordering policies for non-instantaneous deteriorating items with price dependent demand and two storage facilities. *Annals of Operations Research*, *248*(1-2), 253-280.

Khakzad, A., & Gholamian, M. R. (2020). The effect of inspection on deterioration rate: An inventory model for deteriorating items with advanced payment. *Journal of Cleaner Production*, *254*, 120117. https://doi.org/10.1016/j.jclepro.2020.120117

Khan, M. A. A., Shaikh, A. A., Konstantaras, I., Bhunia, A. K., & Cárdenas-Barrón, L. E. (2020). Inventory models for perishable items with advanced payment, linearly time-dependent holding cost and demand dependent on advertisement and selling price. *International Journal of Production Economics*, *230*, 107804. https://doi.org/10.1016/j.ijpe.2020.107804

Khan, M. A. A., Shaikh, A. A., Panda, G. C., Bhunia, A. K., & Konstantaras, I. (2020). Non-instantaneous deterioration effect in ordering decisions for a two-warehouse inventory system under advance payment and backlogging. *Annals of Operations Research*, *289*, 243-275.

Khan, M. A. A., Shaikh, A. A., Panda, G. C., & Konstantaras, I. (2019). Two-warehouse inventory model for deteriorating items with partial backlogging and advance payment scheme. *RAIRO-Operations Research*, *53*(5), 1691-1708. https://doi.org/10.1051/ro/2018093

Kumar, P., & Keerthika, P. S. (2018). An inventory model with variable holding cost and partial backlogging under interval uncertainty: Global criteria method. *International Journal of Mechanical Engineering and Technology*, *9*(11), 1567-1578.

Lashgari, M., Taleizadeh, A. A., & Sadjadi, S. J. (2018). Ordering policies for non-instantaneous deteriorating items under hybrid partial prepayment, partial trade credit and partial backordering. *Journal of the Operational Research Society*, *69*(8), 1167-1196. https://doi.org/10.1080/01605682.2017.1390524

Lee, Y. P., & Dye, C. Y. (2012). An inventory model for deteriorating items under stock-dependent demand and controllable deterioration rate. *Computers & Industrial Engineering*, *63*(2), 474-482. https://doi.org/10.1016/j.cie.2012.04.006

Liao, J. J., Huang, K. N., Chung, K. J., Lin, S. D., Chuang, S. T., & Srivastava, H. M. (2020). Optimal ordering policy in an economic order quantity (EOQ) model for non-instantaneous deteriorating items with defective quality and permissible delay in payments. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, *114*(1), 1-26.

Mahato, C., & Mahata, G. C. (2021). Optimal inventory policies for deteriorating items with expiration date and dynamic demand under two-level trade credit. *OPSEARCH*, …, 1-24.

Maihami, R., Ghelehkhondabi, I., & Ahmadi, E. (2021). Pricing and inventory planning for non-instantaneous deteriorating products with greening investment: A case study in beef industry. *Journal of Cleaner Production*, 126368. https://doi.org/10.1016/j.jclepro.2021.126368

Maiti, A. K., Maiti, M. K., & Maiti, M. (2009). Inventory model with stochastic lead-time and price dependent demand incorporating advance payment. *Applied Mathematical Modelling*, *33*(5), 2433-2443. https://doi.org/10.1016/j.apm.2008.07.024

Mashud, A. H. M., Hasan, M. R., Wee, H. M., & Daryanto, Y. (2019). Non-instantaneous deteriorating inventory model under the joined effect of trade-credit, preservation technology and advertisement policy. *Kybernetes*, *49*(6), 1645-1674. https://doi.org/10.1108/K-05-2019-0357

Mashud, A. H. M., Roy, D., Daryanto, Y., & Wee, H. M. (2021). Joint pricing deteriorating inventory model considering product life cycle and advance payment with a discount facility. *RAIRO-Operations Research*, *55*, 1069-1088. https://doi.org/10.1051/ro/2020106

Mashud, A. H. M., Wee, H. M., Sarkar, B., & Li, Y. H. C. (2020). A sustainable inventory system with the advanced payment policy and trade-credit strategy for a two-warehouse inventory system. *Kybernetes*, …. https://doi.org/10.1108/K-01-2020-0052

Mishra, U., Wu, J. Z., & Sarkar, B. (2021). Optimum sustainable inventory management with backorder and deterioration under controllable carbon emissions. *Journal of Cleaner Production*, *279*, 123699. https://doi.org/10.1016/j.jclepro.2020.123699

Musa, A., & Sani, B. (2012). Inventory ordering policies of delayed deteriorating items under permissible delay in payments. *International Journal of Production Economics*, *136*(1), 75-83. https://doi.org/10.1016/j.ijpe.2011.09.013

Papachristos, S., & Skouri, K. (2000). An optimal replenishment policy for deteriorating items with time-varying demand and partial–exponential type–backlogging. *Operations Research Letters*, *27*(4), 175-184. https://doi.org/10.1016/S0167-6377(00)00044-4

Rahman, M. S., Khan, M. A. A., Halim, M. A., Nofal, T. A., Shaikh, A. A., & Mahmoud, E. E. (2021). Hybrid price and stock dependent inventory model for perishable goods with advance payment related discount facilities under preservation technology. *Alexandria Engineering Journal*, *60*(3), 3455-3465. https://doi.org/10.1016/j.aej.2021.01.045

Rajan, R. S., & Uthayakumar, R. (2017). EOQ model for time dependent demand and exponentially increasing holding cost under permissible delay in payment with complete backlogging. *International Journal of Applied and Computational Mathematics*, *3*(2), 471-487.

Rezagholifam, M., Sadjadi, S. J., Heydari, M., & Karimi, M. (2020). Optimal pricing and ordering strategy for non-instantaneous deteriorating items with price and stock sensitive demand and capacity constraint. *International Journal of Systems Science: Operations & Logistics*, …, 1-12. https://doi.org/10.1080/23302674.2020.1833259

Sen, N., Saha, S., & Nath, B. (2017). Inventory model with price dependent demand under permissible delay in payment. *Asian Journal of Mathematics and Computer Research*, *20*(1), 1-12.

Shah, N. H., & Naik, M. K. (2020). Inventory policies for deteriorating items with time-price backlog dependent demand. *International Journal of Systems Science: Operations & Logistics*, *7*(1), 76-89. https://doi.org/10.1080/23302674.2018.1506062

Shaikh, A. A., Das, S. C., Bhunia, A. K., Panda, G. C., & Khan, M. A. A. (2019). A two-warehouse EOQ model with interval-valued inventory cost and advance payment for deteriorating item under particle swarm optimization. *Soft Computing*, *23*(24), 13531-13546.

Skouri, K., Konstantaras, I., Papachristos, S., & Teng, J. T. (2011). Supply chain models for deteriorating products with ramp type demand rate under permissible delay in payments. *Expert Systems with Applications*, *38*(12), 14861-14869. https://doi.org/10.1016/j.eswa.2011.05.061

Soni, H. N., & Suthar, D. N. (2020). Optimal pricing and replenishment policy for non-instantaneous deteriorating items with varying rate of demand and partial backlogging. *OPSEARCH*, *57*, 986-1021.

Sundararajan, R., & Uthayakumar, R. (2018). Optimal pricing and replenishment policies for instantaneous deteriorating items with backlogging and permissible delay in payment under inflation. *American Journal of Mathematical and Management Sciences*, *37*(4), 307-323. https://doi.org/10.1080/01966324.2017.1422202

Taleizadeh, A. A. (2014a). An economic order quantity model for deteriorating item in a purchasing system with multiple prepayments. *Applied Mathematical Modelling*, *38*(23), 5357-5366. https://doi.org/10.1016/j.apm.2014.02.014

Taleizadeh, A. A. (2014b). An EOQ model with partial backordering and advance payments for an evaporating item. *International Journal of Production Economics*, *155*, 185-193. https://doi.org/10.1016/j.ijpe.2014.01.023

Taleizadeh, A. A., & Nematollahi, M. (2014). An inventory control problem for deteriorating items with back-ordering and financial considerations. *Applied Mathematical Modelling*, *38*(1), 93-109. https://doi.org/10.1016/j.apm.2013.05.065

Taleizadeh, A. A., Niaki, S. T. A., & Barzinpour, F. (2011). Multiple-buyer multiple-vendor multi-product multi-constraint supply chain problem with stochastic demand and variable lead-time: A harmony search algorithm. *Applied Mathematics and Computation*, *217*(22), 9234-9253. https://doi.org/10.1016/j.amc.2011.04.001

Taleizadeh, A. A., Pentico, D. W., Jabalameli, M. S., & Aryanezhad, M. (2013). An economic order quantity model with multiple partial prepayments and partial backordering. *Mathematical and Computer Modelling*, *57*(3-4), 311-323. https://doi.org/10.1016/j.mcm.2012.07.002

Taleizadeh, A. A., Tavakoli, S., & San-José, L. A. (2018). A lot sizing model with advance payment and planned backordering. *Annals of Operations Research*, *271*(2), 1001-1022.

Teng, J. T., Cárdenas-Barrón, L. E., Chang, H. J., Wu, J., & Hu, Y. (2016). Inventory lot-size policies for deteriorating items with expiration dates and advance payments. *Applied Mathematical Modelling*, *40*(19-20), 8605-8616. https://doi.org/10.1016/j.apm.2016.05.022

Udayakumar, R., Geetha, K. V., & Sana, S. S. (2020). Economic ordering policy for non-instantaneous deteriorating items with price and advertisement dependent demand and permissible delay in payment under inflation. *Mathematical Methods in the Applied Sciences*. https://doi.org/10.1002/mma.6594

Viji, G., & Karthikeyan, K. (2018). An economic production quantity model for three levels of production with Weibull distribution deterioration and shortage. *Ain Shams Engineering Journal*, *9*(4), 1481-1487. https://doi.org/10.1016/j.asej.2016.10.006

Wang, W.C., Teng, J.T. & Lou, K.R. (2014). Seller's optimal credit period and cycle time in a supply chain for deteriorating items with maximum lifetime. *European Journal of Operational Research,* 232(2), 315-321. https://doi.org/10.1016/j.ejor.2013.06.027

Wu, J., Skouri, K., Teng, J. T., & Ouyang, L. Y. (2014). A note on "optimal replenishment policies for non-instantaneous deteriorating items with price and stock sensitive demand under permissible delay in payment." *International Journal of Production Economics*, *155*, 324-329. https://doi.org/10.1016/j.ijpe.2013.12.017

Wu, J., Teng, J. T., & Chan, Y. L. (2018). Inventory policies for perishable products with expiration dates and advance-cash-credit payment schemes. *International Journal of Systems Science: Operations & Logistics*, *5*(4), 310-326.

Wu, K. S., Ouyang, L. Y., & Yang, C. T. (2006). An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. *International Journal of Production Economics*, *101*(2), 369-384. https://doi.org/10.1016/j.ijpe.2005.01.010

Zhang, Q., Tsao, Y. C., & Chen, T. H. (2014). Economic order quantity under advance payment. *Applied Mathematical Modelling*, *38*(24), 5910-5921. https://doi.org/10.1016/j.apm.2014.04.040

Zia, N. P., & Taleizadeh, A. A. (2015). A lot-sizing model with backordering under hybrid linked-to-order multiple advance payments and delayed payment. *Transportation Research Part E: Logistics and Transportation Review*, *82*, 19-37. https://doi.org/10.1016/j.tre.2015.07.008

Appendix A. Proof of Theorem 1

Using equation (18) for each F, $f(T)$ is defined as follow:

$$f(T)=\begin{pmatrix} A+i_1\alpha C_p D\begin{bmatrix} FT+\dfrac{\theta F^2T^2}{2}-\theta FTt_d \\ +\dfrac{\theta t_d{}^2}{2}+\dfrac{1}{\delta}\ln\left[1+\delta(1-F)T\right] \end{bmatrix}\dfrac{(n+1)}{2n}L \\ +C_pD\begin{bmatrix} FT+\dfrac{\theta F^2T^2}{2}-\theta FTt_d \\ +\dfrac{\theta t_d{}^2}{2}+\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta} \end{bmatrix}+C_hD\begin{pmatrix} t_d{}^2-FTt_d \\ +\dfrac{1}{2}F^2T^2 \end{pmatrix} \\ +\dfrac{C_bD}{\delta}\begin{pmatrix} (1-F)T \\ -\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta} \end{pmatrix}+GD\begin{pmatrix} (1-F)T \\ -\dfrac{\ln\left[1+\delta(1-F)T\right]}{\delta} \end{pmatrix} \end{pmatrix} \tag{A1}$$

And $g(T) = T > 0$.

Then $ATC(F,T) = f(T)/g(T)$. We take the partial derivatives of $f(T)$ by regarding T as the following:

$$f'(T)=\begin{pmatrix} \left(i_1\alpha C_pD\dfrac{(n+1)}{2n}L+C_pD\right)\begin{pmatrix} F+\theta F^2T-\theta Ft_d \\ +\dfrac{(1-F)}{1+\delta(1-F)T} \end{pmatrix} \\ +C_hD\left(-Ft_d+F^2T\right)+\left(\dfrac{C_bD}{\delta}+GD\right)\begin{pmatrix} (1-F) \\ -\dfrac{(1-F)}{1+\delta(1-F)T} \end{pmatrix} \end{pmatrix} \tag{A2}$$

and

$$f''(T)=\begin{pmatrix} \left(i_1\alpha C_pD\dfrac{(n+1)}{2n}L+C_pD\right)\left(\theta F^2\right)+C_hDF^2 \\ +\left(\dfrac{\delta(1-F)^2}{\left(1+\delta(1-F)T\right)^2}\right) \\ \left(\dfrac{C_bD}{\delta}+GD-i_1\alpha C_pD\dfrac{(n+1)}{2n}L-C_pD\right) \end{pmatrix} \tag{A3}$$

Therefore, $ATC(F,T) = f(T)/g(T)$ is entirely pseudo-convex in T if $\left(\frac{C_b}{\delta}+G\right)-C_p\left(i_1\alpha\frac{(n+1)}{2n}L+1\right)\geq 0$, because by applying this condition $D[f''(T)\geq 0]$, and the condition of convexity of the function $f(T)$ is established and completes the proof of (I) into Theorem 1.

To prove the second and third parts of Theorem 1, we take the first-order derivative of $ATC(F,T)$ with respect to T, and rearranging in terms, we have:

$$\frac{\partial ATC(F,T)}{\partial T}=\begin{bmatrix} -\frac{A}{T^2}+\left(i_1\alpha C_p D\frac{(n+1)}{2n}L+C_pD\right)\begin{pmatrix}\frac{\theta F^2}{2}-\frac{\theta t_d{}^2}{2T^2}\\ -\frac{\ln\left[1+\delta(1-F)T\right]}{\delta T^2}\\ +\frac{(1-F)}{T\left(1+\delta(1-F)T\right)}\end{pmatrix}\\ +C_hD(-\frac{t_d{}^2}{T^2}+\frac{1}{2}F^2)+\left(\frac{C_bD}{\delta}+GD\right)\begin{pmatrix}\frac{\ln\left[1+\delta(1-F)T\right]}{\delta T^2}\\ -\frac{(1-F)}{T\left(1+\delta(1-F)T\right)}\end{pmatrix}\end{bmatrix} \quad (A4)$$

Now to applying the Mean value theorem in the interval $[t_d,\infty]$, we first form the following limit and then calculate the value of the first derivative of $ATC(F,T)$ at point of t_d.

$$\lim_{T\to\infty}(\frac{\partial ATC(F,T)}{\partial T})$$
$$=\begin{bmatrix} -\frac{A}{T^2}+\left(i_1\alpha C_p D\frac{(n+1)}{2n}L+C_pD\right)\begin{pmatrix}\frac{\theta F^2}{2}-\frac{\theta t_d{}^2}{2T^2}\\ -\frac{\ln\left[1+\delta(1-F)T\right]}{\delta T^2}\\ +\frac{(1-F)}{T\left(1+\delta(1-F)T\right)}\end{pmatrix}\\ +C_hD(-\frac{t_d{}^2}{T^2}+\frac{1}{2}F^2)+\left(\frac{C_bD}{\delta}+GD\right)\begin{pmatrix}\frac{\ln\left[1+\delta(1-F)T\right]}{\delta T^2}\\ -\frac{(1-F)}{T\left(1+\delta(1-F)T\right)}\end{pmatrix}\end{bmatrix}=\infty \quad (A5)$$

Then by applying Equation A4 and putting $T=t_d$, we know:

$$ATC'(F,t_d)=\left(i_1\alpha C_p D\frac{(n+1)}{2n}L+C_pD\right)\begin{pmatrix}\frac{\theta F^2}{2}-\frac{\ln\left[1+\delta(1-F)t_d\right]}{\delta t_d{}^2}\\ -\frac{\theta}{2}+\frac{(1-F)}{t_d\left(1+\delta(1-F)t_d\right)}\end{pmatrix}$$
$$+C_hD(\frac{1}{2}F^2-1)+\left(\frac{C_bD}{\delta}+GD\right)\begin{pmatrix}\frac{\ln\left[1+\delta(1-F)t_d\right]}{\delta t_d{}^2}\\ -\frac{(1-F)}{t_d\left(1+\delta(1-F)t_d\right)}\end{pmatrix}-\frac{A}{t_d{}^2}=L_1 \quad (A6)$$

If $L_1 \geq 0$, using the Mean value theorem and applying (A5) and (A6), there is a unique $T \in [t_d, \infty]$, such that $ATC\ (F,T)$ can be minimized. This is also considered as the proof of (II) into Theorem 1.

If $L_1 < 0$, for all $T \in (0, t_d]$, $ATC'(F,T) < 0$. So $ATC\ (F,T)$ is decreasing in T and would be minimized at $T = t_d$. This is considered as the proof of (III) into Theorem 1 then.

Appendix B. Proof of Theorem 2

For each T, by taking the first and second partial derivations of ATC with respect to F, we define:

$$\frac{\partial ATC(F,T)}{\partial F} = \begin{pmatrix} \left(i_1 \alpha C_p D\frac{(n+1)}{2n}L + C_p D\right)\left(1+\theta FT - \theta t_d - \frac{1}{1+\delta(1-F)T}\right) \\ + C_h D(-t_d + FT) + \left(\frac{C_b D}{\delta} + GD\right)\left(-1+\frac{1}{1+\delta(1-F)T}\right) \end{pmatrix} \quad \text{(B1)}$$

And

$$\frac{\partial^2 ATC(T,F)}{\partial F^2} = DT\begin{bmatrix} \left(\frac{C_b}{\delta} + G - \left(i_1 \alpha C_p \frac{(n+1)}{2n}L + C_p\right)\right)\left(\frac{\delta}{(1+\delta(1-F)T)^2}\right) \\ +\theta\left(i_1 \alpha C_p \frac{(n+1)}{2n}L + C_p\right) + C_h \end{bmatrix} \quad \text{(B2)}$$

Consequently, $ATC(F,T)$ is convex in F if $\left(\frac{C_b}{\delta} + G\right) - C_p\left(i_1 \alpha \frac{(n+1)}{2n}L + 1\right) \geq 0$.

Applying Equation B1 in the interval $[0,1]$, we have:

$$ATC'(T, F=1) = D(T - t_d)\left[\theta C_p\left(i_1 \alpha \frac{(n+1)}{2n}L + 1\right) + C_h\right] = L_3 \quad \text{(B3)}$$

If $L_2 > 0$ and $L_3 \geq 0$, then $ATC'(F) > 0$ for whole $F \in (0,1]$ and $ATC(F,T)$ is incrementing in F; hence, it is minimized at $F = 0$.

If $L_2 > 0$ and $L_3 < 0$, then by using the Mean value theorem, $F^* \in (0,1)$ as $ATC(F,T)$ is minimized.

If $L_2 < 0$ and $L_3 \leq 0$, then $ATC'(F) < 0$ for whole $F \in [0,1]$ and $ATC(F,T)$ is reducing in F; hence, it would be minimized at $F^* = 1$. This is also considered as the proof into Theorem 2.