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RESEARCH PAPER

CAT Bond Pricing in Uncertain Environment

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Abstract
Catastrophe bonds are among the essential instruments in providing a financial hedge for insurance companies and their policyholders. Catastrophic events are rare, and the shortage of data turns using probability theory indefensible. On the other hand, uncertainty theory is a reliable alternative to deal with these kinds of indeterminacies. We model the problem of pricing catastrophe bonds as an uncertain optimization problem where the maximization of the cedent insurance company's profit is constrained to the uncertain measure of ruin defined for the investors. Consequently, one could provide a tradeoff between being profitable for the ceding company and having reasonable protection for the investors. A solution to the optimization problem will be considered as the spread over the LIBOR, leading to a complete determination of the bond price. The results suggest the practicality of the model, especially the application of uncertainty theory in pricing catastrophe bonds. Finally, the uncertain ruin index is calculated for a real-world problem, and the results are compared with those obtained by probability theory.

Keywords: CAT bond, insurance-linked securities pricing, uncertainty theory, uncertain programming, uncertain process

Introduction

Designing catastrophe bonds (shortly known as CAT bonds) and their pricing in a very uncertain environment is the main concern of financial institutes. Though there are several theories dealing with the indeterminacy of situations, one may conclude to employ probability theory as the main paradigm since of its axiomatic framework and victorious history in solving many problems. However, probability theory requires sufficient reliable data because of its very nature for approximating proper probability distribution functions. In the absence of this requirement, the problem must be dealt with using some indeterministic theory designed for these specific situations. Here, we first describe the necessity for designing the CAT bonds before involving in the mathematical modeling. The use of uncertainty theory instead of probability theory would be inferred after presenting a suitable background about the bond structure. We hope it fills the gap in this area and provides a sensible justification for our proposed approach.

The term "uncertainty" is a general concept in the scientific field, which defines the environment in which the future outcome cannot be fully predicted for a particular event. Here, it refers to the specific notion introduced in the uncertainty theory, suggested and mathematically axiomatized by Liu (2007). This theory works well in dealing with the problems to deal with which an expert and his opinion quantify the uncertain situation. Especially in some highly unpredictable phenomena, one has just the option of relying on the expert opinion

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about the potential outcome. Such situations reveal themselves in diverse fields, from the outbreak of a fresh contagious disease like COVID-19 to natural disasters like floods, tornadoes, and volcanos. Even though some data may exist, such events' complicated and uncovered nature implies inviting some experts to quantify their possible effects on the general economy and its different sections. In this study, we project this uncertainty on a financial instrument called CAT bonds and analyze its effectiveness in the decision-making process. The main objective of this study is to provide a methodology for pricing CAT bonds using the tools provided by the uncertainty theory.

Natural disasters such as earthquakes and floods would simultaneously result in dramatic damages and massive financial losses to civilians and financial institutes. The main characteristics of these events are their low frequency and high severity, making it almost impossible to predict their aftermath properly, and consequently complicate their accurate hedging. In actuarial science, such events are known as catastrophic events. The uncertainty is tied to a catastrophe, mainly to its potential occurrence and scale. Here, we do not aim to model other aspects of the insurance business that also have some sort of uncertainty in their nature. Moral hazard and associated behavior of the customers as well as the implications enforced by the governmental organizations are some other types of uncertainties. While these features can also be incorporated into the model, we do not address them here due to the complexity resulting from these considerations. See Tajeddini and Trueman (2016) and Tajeddini and Mueller (2019) for more cases of these kinds of uncertainties.

Due to these particularities of catastrophic events, insurance companies are highly exposed to such conditions; hence, they usually cannot fully or partially meet their commitments as a result of the overburdening claims of these events. To deal with this challenge and avoid undesired failure, insurance companies utilize alternatives, e.g., reinsurance, to transfer fully or share the associated risks partially to another party. However, the reinsurance companies may not be powerful enough to cover the commitments; thus, more reliable protection would eventually be necessary. CAT bonds have been a trustable instrument to transfer such an unbearable risk to the capital market, the place where the risk can be properly hedged. The next few paragraphs are devoted to defining the structure and the role of different parties involved in the CAT bond's performance in the capital market.

A CAT bond is insurance-linked security designed to deliver some share of predefined risks to the capital market. Its risk is attached to the hazard of severe natural events (e.g., hurricanes, earthquakes, volcanoes, and floods) or human-made disasters (e.g., nuclear power plant failure, wildfires, and mining accidents). These bonds empower insurance and reinsurance companies by enabling them to meet capital requirements obliged by the regulators. Moreover, they provide the flexibility required for companies to endure more risks and engage in their very business (Krutov, 2010). The basic frameworks of the CAT bonds are designed to be flexible over different kinds of events and the involved parties' concerns.

A CAT bond intrinsically includes several specific triggers for activation. Depending on the bond's terms, the activation would partially or completely diminish the investors' principal. Because of this inherent credit risk attached to a CAT bond, investors expect a higher return rate than other securities such as corporate bonds (Bodoff & Gan, 2009). This extra reward makes the bond more interesting, while it comes with an unexpected higher risk. Hence, these bond features demand very careful treatment in the pricing process as it is counted as the main feature in introducing to the capital market. Furthermore, the CAT bond can include a coupon payment or be a zero-coupon bond, making it more flexible and optional. In the following, a standard version of the bond is described in detail. Note that more details would be attached to such a model for a specific event, and the components would vary based on agreements between the parties.

Structure of a CAT Bond

A CAT bond generally has three players: the sponsor (cedent) intends to transfer the corresponding risk to the capital market, the Special Purpose Vehicle (SPV) that issues the bond, and the investors. They admit this risk against gaining potential higher profit. The sponsor can be an insurance or reinsurance company or any entity looking for such protections. SPV involves in the contract with the sponsor to issue its risk through the bonds. It collects the premium from the sponsor and proceeds from the sale to a collateral account. The sponsor has no claim on the assets, and SPV is considered a bankruptcy-remote body. Conditioning on no catastrophic event mentioned in the bond, investors might receive a coupon periodically. Another option would be collecting the whole amount at the end of the term, including the initial premium and additional profit calculated using an agreed interest rate. The interest rate is mostly based on LIBOR added with an extra spread to cover a potential occurrence risk of the catastrophic event. If the mentioned catastrophe in the contract happens during the bond term, the investors lose a part of or the whole principal. SPV then compensates the sponsor against the loss. A simplified scheme of this structure is depicted in Figure 1. As pointed out in Bodoff and Gan (2009), the coupon is a sum-up of LIBOR and a spread. Since the swap counterpart contract essentially fixes the LIBOR, the only parameter to calculate in the pricing process is the spread. Spread, on the other hand, is perfectly related to the collected premium from the sponsor. Therefore, identifying a proper spread is a challenging issue in the field.

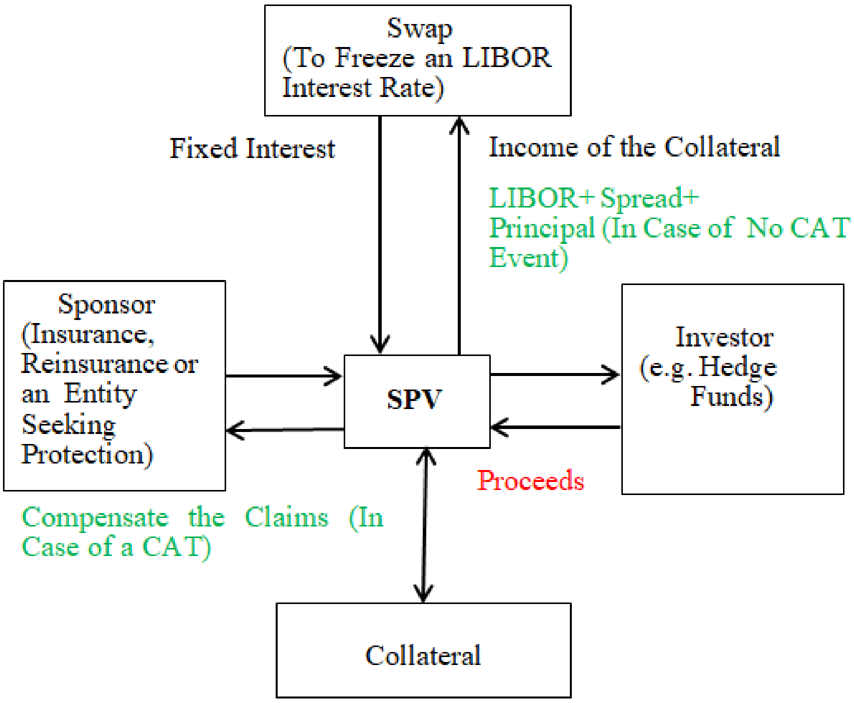

Figure 1. A Simplified Structure of a CAT Bond

It should be noted that almost all model elements in expressing catastrophic events intrinsically have indeterministic nature since they are among the most unpredictable phenomena in nature. For example, the number and amount of losses in a particular period are unknown because of the underwritings associated with such an event. By overlooking all uncertainties, deterministic models do not appreciate the problem’s conditions and cannot be reliably adopted for a real-world instance. Different approaches are considered to handle the indeterminacy of the problem. Although several studies have been carried out in CAT bond pricing using the probability theory, there are still some essential challenges in the field. Note that considering several scenarios for the possible outcome is an option. Probabilistic models define a structural procedure to discover a pattern out of the data. To this end, they may need sufficiently trustable data, which is not accessible in most cases. Therefore, some existing data on similar events are utilized to generate more likely scenarios and associate a probability distribution to each of them. However, there is no specific modeling tool for predicting the potential outcome of all possible scenarios.

Moreover, estimating the parameters and consequently pricing the bond, which needs costly calculations, are accompanied by several simplifications in the model. More importantly, as a probabilistic model's basic ingredients, suitably collecting data and filtering the samples are not always attainable. In summary, probabilistic elements are not properly constructed in these situations, and generating exact distribution functions is almost impossible. Consequently, applying a probabilistic model would produce a solution that does not accurately interpret reality but becomes an extremely simplified version.

In such a situation, when data are sparse and not reliable, another option for the production of some information is to invite some experts in the field, e.g., seismologists and economists, to contribute their opinions on the possible quantified outcome. They might be the only available raw data considered through the whole CAT bond modeling and ultimate pricing task in many situations. It means that the whole modeling procedure and inference process is based on expert reasoning, and we need a strong methodology for the inference process.

To model human reasoning in a mathematical framework, Liu (2007) proposed a nondeterministic approach, referred to as uncertainty theory. It is a well-structured construction to handle problems when there is not enough data, and one must either work with a very small sample or rely on the expert's opinion. Uncertainty theory has been designed to deal with problems in which belief is the benchmark of modeling, and basic structures such as distributions are made based on the quantitative opinion of some experts in the field. It benefits from the measure theory's axiomatics for incorporating the belief degree in analyzing uncertain events. One of the main differences between uncertainty theory and probability theory is how the independence of events, and later of uncertain variables, is defined (Liu, 2017). Referring to the nature of catastrophic events for which no data exists or existing data cannot be applied to a fresh event, a reasonable choice is to construct the whole modeling process on uncertainty theory. Especially, it can be inferred that CAT bonds pricing is more adjustable to be studied through uncertainty theory rather than probability theory. We refer the interested reader to Liu (2012, 2017) for a good discussion about the logic and arguments behind this theory's axioms and the convincing logic behind the theory itself.

This paper studies the CAT bond pricing problem from the uncertainty theory standpoint and assumes that an expert opinion is available for some of its uncertain factors. We generally consider rare events where relevant data are scarce and mostly not trustable. Using simple and practical calculations in this theory, we model an insurance company's losses and gains as an optimization problem to identify the optimal amount of spread beneficial to both the sponsor and the investor, using the capabilities of uncertainty theory. To the best of our knowledge, this is the first study of pricing CAT bonds using this theory; we believe our work would be a benchmark for further research projects and interesting findings in this very practical subject and similar situations.

The rest of the paper is organized as follows. We first mention some findings from the stochastic point of view in Section 2. Then, we briefly review uncertainty theory and mention some of the necessary concepts we use in our model in Section 3. The model is proposed in Section 3.1, including definitions of the uncertain process describing the CAT bond and devising an optimization model. Section 4 is devoted to a prototype instance in addition to a real-world sample. The proposed model's tractability and practicability are also examined in these examples. Finally, in Section 5, some concluding remarks are mentioned, and further research directions are suggested.

Literature Review

Hurricane Andrew in 1992 was the first catastrophic event whose risk was adjusted as a call option spread and introduced to the capital market. It was withdrawn later due to the lack of trading volume. In 1994, Hannover Re Group designed a CAT bond and issued USD 85 million of risk to the market. Covering earthquake risk in Tokyo was the subject of a CAT bond in 1999 (Cummins, 2012). Over recent years, the trust in CAT bonds has been increased, and the financial market is more engaged toward introducing other standard cases of these insurance-linked securities. As reported by Artemis deal directory, just in 2020, more than USD 14 billion insurance-linked securities have been issued, with more than 44 billion outstanding CAT bonds taking the dominant position in this market share (Artemis, 2020).

Pricing is the most important task in proposing a CAT bond to the capital market. It is always challenging for actuaries, and numerous studies have been devoted to this problem. Existing approaches can be categorized as economic and contingent claims methods. In the former, the historical data are used to fit the pricing parameters into a regression function. The model applies the produced function as the basis in the future pricing process of afresh-offered bonds. Lane (2000) used a power function for the regression, where the probability of the first loss and the conditional expected loss are the main drivers of the model. Lane and Mahul (2008) then employed a substantial volume of data about the previously issued bonds to provide a linear function containing some involved parameters such as the expected value, peril, reinsurance cycle, and rating. These findings are useful when the market has a reasonable history of trading CAT bonds, whereas it is barely practical for markets with no such rich history. Moreover, each specific event demands its own data, and defined CAT bonds cannot be very valuable for the other catastrophic events.

Similar results have been obtained by taking more parameters into account (Lei et al., 2008). Bodoff and Gan (2009) generalized these sorts of models to make a more trustful regression function, where identifying the spread of the CAT bond was the objective of the regression. Several perils were combined to extend the idea one more step toward a complete model, considering almost all of the parameters involved. For more details, we refer the interested reader to Braun (2016) and Wang (2002), and the review paper by Galeotti et al. (2013).

In the contingent approach toward pricing, the policy is similar to those used in defaultable bonds introduced by Duffie and Singleton (1999). The market's response is not considered in this approach, but the potential claims' future and prediction are the main components. The price of a CAT bond is the basis of modeling rather than its spread over the interest rate. Here, we mention some of the pricing methods related to this approach. Equilibrium modeling is another approach where the involved individuals, insurance company, and investors of the CAT bond are the main units of analysis. Utility and concerns about the involvement are incorporated into the model, and the final solution satisfies the parties as an equilibrium point. This method was pioneered by Aase (1999) and Cox and Pedersen (2000). Vaugirard (2003) used a jump-diffusion process to model the dynamics of the underlying index. In another study by Zimbidis et al. (2007), the so-called historical probability measure was the basis of the methodology. Nowak and Romaniuk (2013) applied the results in Vaugirard (2003) to extend it to an improved and more general model. In their generalized model, the Vasicek model (1977) was applied for the interest rate dynamics in the pricing of both the CAT bond and the interest rate modeled as a Cox-Ingersoll-Ross process. Using the idea proposed in Duffie and

Singleton (1999) and Zhou (1997), Baryshnikov et al. (2001) priced the CAT bonds utilizing an auxiliary bond, referred to as threshold bond.

Stochastic optimization has also been applied in catastrophe risk modeling (Ermoliev et al., 2000). The authors implemented the obtained model to calculate some parameters such as ruin probability as the key risk measure in calculating the insurance premium. Later, the optimization methods have been applied to optimally combine the reinsurance and the CAT bond to benefit the sponsor in proposing an interesting price to the market (Lakdawalla & Zanjani, 2012).

As an alternative in nondeterministic modeling, fuzzy theory was also applied in the CAT bond pricing problem (Nowak & Romaniuk, 2013, 2014, 2017). Many arguments are on fuzzy theory's efficiency over belief-based problems (Henkind & Harrison, 1988; Kickert, 1979; Liu, 2017). Here, we mention a few of its disadvantages in dealing with problems like catastrophe risk measuring. First, recall that there are many choices for membership function, with no clear reasoning in utilizing the one with the most efficiency. Detecting a suitable and problem-oriented membership function is always restricting greatly. Moreover, computational operators in this theory are either too complicated or too simple. While those complicated ones are more sensible in real-world problems, those with simple constructions do not perform better in fitting practical problems. Moreover, the flexibility of the choices in different parts of this theory makes more trouble than easing the model and finding the right solution (Henkind & Harrison, 1988).

Basics of Uncertainty Theory

Over a nonempty set Γ, consider σ-algebra L. Each element $\Lambda \in L$ is called an event. A set function $\mathcal{M}$ from L to $[0,1]$ satisfying the following axioms is referred to as an uncertain measure (Liu, 2007).

Axiom 1 (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2 (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$,

$$\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\} \tag{1}$$

The triplet $(\Gamma, L, \mathcal{M})$ is referred to as an uncertainty space. The fourth axiom, introduced for the product of events, distinguishes the probability theory from uncertainty theory.

Axiom 4 *(Product Axiom) (Liu, 2009) Let* $(\Gamma_k, L_k, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1,2,\ldots$ *The product uncertain measure* $\mathcal{M}$ *is a measure on product* σ*-algebra* $L_1 \times L_2 \times \cdots \times L_n$ *satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}. \tag{2}$$

An uncertain variable is defined to quantitatively deal with phenomena in uncertainty theory (Liu, 2007). It is a measurable function from an uncertain space $(\Gamma, L, \mathcal{M})$ to the set of real numbers, in which for any Borel set B, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

Uncertainty distribution of an uncertain variable ξ is defined as (Liu, 2007)

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \qquad \forall x \in \Re. \tag{3}$$

An uncertainty distribution $\Phi(x)$ is called regular if it is a continuous and strictly increasing function w.r.t. x for all $0 \le \Phi(x) \le 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \qquad \lim_{x \to +\infty} \Phi(x) = 1. \tag{4}$$

Satisfying this property lets one to bypass the computation burden of rigorous uncertainty distributions by using their inverses, making the solution methods more tractable. Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. The function $\Phi^{-1}(\alpha)$ is called inverse uncertainty distribution of ξ.

Some uncertain variables have been defined in uncertainty theory. The most typical one is the linear uncertain variable recognized by its uncertainty distribution as

$$\Phi(x) = \begin{cases} 0 & \text{if } x \le a, \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{if } x \ge b, \end{cases} \tag{5}$$

and denoted by $\mathcal{L}(a,b)$ where a and b are real numbers with $a < b$. Its inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b. \tag{6}$$

An uncertain variable ξ is called normal and denoted by $\mathcal{N}(e,\sigma)$ (Liu, 2007), if it possesses the uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \in \Re, \tag{7}$$

where e and σ are real numbers with $\sigma > 0$. The important practical one in our study is the lognormal uncertain variable. Its uncertainty distribution reads as

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-\ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \ge 0. \tag{8}$$

Its inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = \exp\left(e + \sigma\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right). \tag{9}$$

Uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent if (Liu, 2009)

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}, \tag{10}$$

for arbitrary Borel sets $B_1, B_2, \ldots, B_n$.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing w.r.t. $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing w.r.t. $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then f is an uncertain variable with the uncertainty distribution (Liu, 2010)

$$\Psi(x)=\sup_{f(x_1,x_2,\ldots,x_n)=x}\left(\min_{1\le i\le m}\Phi_i(x_i)\wedge\min_{m+1\le i\le n}(1-\Phi(x_i))\right). \tag{11}$$

Theorem 3.1 (Liu, 2007) *Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. Furthermore, let $f(\xi_1,\xi_2,\ldots,\xi_n)$ be strictly increasing w.r.t. $\xi_1,\xi_2,\ldots,\xi_m$ and strictly decreasing w.r.t. $\xi_{m+1},\xi_{m+2},\ldots,\xi_n$. Then, f is an uncertain variable with the inverse uncertainty distribution*

$$\Psi^{-1}(\alpha)=f(\Phi^{-1}(\alpha),\ldots,\Phi^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha)). \tag{12}$$

Theorem 3.2 (Liu, 2010a) Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\ldots,\Phi_n$, respectively. If $f(\xi_1,\ldots,\xi_n)$ is strictly increasing w.r.t. $\xi_1,\xi_2,\ldots,\xi_m$ and strictly decreasing w.r.t. $\xi_{m+1},\xi_{m+2},\ldots,\xi_n$, then

$$\mathcal{M}\{f(\xi_1,\xi_2,\ldots,\xi_n)\le 0\}, \tag{13}$$

is the root α of the equation

$$f(\Phi_1^{-1}(\alpha),\ldots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha))=0. \tag{14}$$

If $f(\Phi_1^{-1}(\alpha),\ldots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha))<0$ for all α, then we set $\alpha=1$; and if $f(\Phi_1^{-1}(\alpha),\ldots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha))>0$ for all α, then we set $\alpha=0$.

The concept of expected value for an uncertain variable ξ is defined as (Liu, 2007)

$$E[\xi]=\int_0^{+\infty}\mathcal{M}\{\xi>r\}dr-\int_{-\infty}^{0}\mathcal{M}\{\xi\le r\}dr \tag{15}$$

provided that at least one of the two integrals is finite. Moreover, if the uncertainty distribution is regular, expected value can be calculated by

$$E[\xi]=\int_0^1\Phi^{-1}(\alpha)d\alpha. \tag{16}$$

Uncertain Process

Let $(\Gamma_k,L_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\ldots$, and T be a totally ordered set (e.g., time). For the sake of simplicity, we use the term "time" for each member of this set. An uncertain process is a function $X_t(\gamma)$ from $T\times(\Gamma_k,L_k,\mathcal{M}_k)$ to the set of real numbers such that $\{X_t\in B\}$ is an event for any Borel set B of real numbers at each time t. An uncertain process X_t has independent increments if

$$X_{t_0},X_{t_1}-X_{t_0},X_{t_2}-X_{t_1},\ldots,X_{t_k}-X_{t_{k-1}}, \tag{17}$$

are independent uncertain variables where t_0 is the initial time and $t_1,t_2,\ldots,t_k$ are any times with $t_0<t_1<t_2<\cdots<t_k$.

Definition 3.1 (Liu, 2008) Let $\xi_1,\xi_2,\ldots$ be iid uncertain interarrival times. Define $S_0=0$ and $S_n=\xi_1+\xi_2+\cdots+\xi_n$ for $n\ge 1$. The uncertain process

$$N_t=\max_{n\ge 0}\{n\mid S_n\le t\}, \tag{18}$$

is called an uncertain renewal process.

Definition 3.2 (Liu, 2010b) Let $\xi_1, \xi_2, \ldots$ be iid uncertain interarrival times. Further, let $\eta_1, \eta_2, \ldots$ be iid uncertain rewards or costs (losses in insurance case) associated with i-th interarrival time ξ_i for $i = 1, 2, \ldots$. Then

$$R_t = \sum_{i=1}^{N_t} \eta_i , \tag{19}$$

is called a renewal reward process, where N_t is the renewal process with uncertain interarrival times $\xi_1, \xi_2, \ldots$.

Theorem 3.3 *(Liu, 2010b) Let R_t be a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \ldots$ and iid uncertain rewards $\eta_1, \eta_2, \ldots$. Assume $(\xi_1, \xi_2, \ldots)$ and $(\eta_1, \eta_2, \ldots)$ are independent uncertain vectors, and those interarrival times and rewards have uncertainty distributions Φ and Ψ, respectively. Then, R_t has uncertainty distribution*

$$\Upsilon_t(x) = \max_{k \ge 0} \left\{ \left(1 - \Phi\left(\frac{t}{k+1} \right) \right) \wedge \Psi\left(\frac{x}{k}\right) \right\}. \tag{20}$$

Theorem 3.4 *(Liu, 2010b) Let R_t be a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \ldots$ and iid uncertain rewards $\eta_1, \eta_2, \ldots$. Assume $(\xi_1, \xi_2, \ldots)$ and $(\eta_1, \eta_2, \ldots)$ are independent uncertain vectors. Then, the reward rate*

$$\lim_{t \to \infty} \frac{R_t}{t} \to \frac{\eta_1}{\xi_1}, \tag{21}$$

in the sense of convergence in distribution.

Theorem 3.5 *(Liu, 2010b) Let R_t be a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \ldots$ and iid uncertain rewards $\eta_1, \eta_2, \ldots$. Assume $(\xi_1, \xi_2, \ldots)$ and $(\eta_1, \eta_2, \ldots)$ are independent uncertain vectors. Then,*

$$\lim_{t \to \infty} \frac{E[R_t]}{t} \to E\left[\frac{\eta_1}{\xi_1} \right]. \tag{22}$$

Definition 3.3 *(Liu, 2013) Let U_t be an insurance risk process responsible for modeling the growth or decline of an insurance company's wealth. Then, the ruin index is defined as the uncertain measure that U_t eventually becomes negative, i.e.,*

$$Ruin = \mathcal{M}\left\{ \inf_{t \ge 0} U_t < 0 \right\}. \tag{23}$$

Optimization Model of the CAT Bond Pricing

This section presents an optimization model for the CAT bond pricing in an uncertain environment. The objective is to maximize the sponsor's wealth while the investor's profit is satisfied in some constraints. First, note that the sponsor prefers to regain as high as possible from its investment, which could be defined as an asymptotic expectation of its wealth. On the other hand, since the investor's earn cannot be negative, we restrict its expected value to be positive at any time. Moreover, to be enough motivation for investment, this earning must be greater than the one in a fixed-interest-rate business.

Recall that all the processes involved in the model are continuous, i.e., there is a chance of a claim occurrence at each time interval with a continuous claim distribution. Thus, applying the ultimate ruin instead of the ruin in a finite time span makes more sense. In conclusion, one can divide that time interval into many potential subintervals and assume that the claims occur in these subintervals. Making it clearer, we consider a model in which the investor and the sponsor's point of view are infinite. To do this end, we assume that the bond term's termination time is divided into many reasonable small subintervals. In this way, the investor's possibility of losing their whole investment does not depend on time. Now, we construct the pricing model step by step to reach the final optimization problem.

Consider a CAT bond where the sponsor wants to transfer the risk of exposure to a high level of claims to the capital market in case of a catastrophic event. An attachment point can identify this high level, denoted here by A. Note that a CAT bond structure has fixed collateral, and it cannot provide the sponsor with an amount more than this number. Therefore, the bond's trigger is defined as a parameter, activated when the sponsor's loss exceeds A. The trigger can be defined as 1_A, and the indicator function is defined as

$$1_A = \begin{cases} 1 & if R_t \geq A \\ 0 & o.w. \end{cases} . \tag{24}$$

when the trigger is activated, the sponsor is compensated accordingly.

Here, we denote the CAT bond as one total amount divided with identical spread among investors according to the number of presented bonds to the market. Thus, without loss of generality, we model the problem with one sponsor and one investor. The risk process of the sponsor is

$$U_t^1 = u + (\pi_I - \pi_C)t - R_t + R_t 1_A. \tag{25}$$

Here u is the initial capital of the sponsor, π_I is the insurance premium collected from insureds, π_C is the spread, is the aggregate loss process in which N_t, and η_i are frequency and severity processes of the claims, respectively. We assume that η_i is iid uncertain variables, and without loss of generality, the severity and frequency of the claims are independent.

We model the investor's wealth as

$$U_t^2 = \Pi(1 + rt) + \pi_C t - R_t 1_A, \tag{26}$$

where Π is the collateral, i.e., the amount of pooling collected from the investor. We also consider a linear interest rate that the investor earns as a part of the return for the investment. Notice that the investor bears a significant risk by exposing to losing risk all or a considerable amount of his principal when the trigger is activated. Therefore, he requires an award yielded through the spread for taking such a risk.

Three main considerations make up the optimization problem:

1. The expected return of the process representing the sponsor's wealth must be as high as possible. Here, we consider it as the model's objective function. It refers to the point that the sponsor makes a profit in the long run.
2. The investor's wealth cannot be negative during the bond's term. Therefore, as a constraint, we assume that the uncertain measure of ruin related to the investor's process is not greater than a small positive number.
3. Because of the potential higher risk that the investor takes by purchasing a CAT bond, he expects a spread that compensates with an amount higher than the asymptotic expectation.

We devise the optimization model as follows.

$$\begin{aligned} &\max_{\pi_C} \quad \lim_{t\to\infty}\frac{E\left[U_t^1\right]}{t} \\ &s.t. \quad Ruin\ Index \le \varepsilon \\ &\lim_{t\to\infty}\frac{E\left[R_t\right]}{t} \le \frac{\pi_C}{(1+r)K} \\ &\pi_C \ge 0. \end{aligned} \tag{27}$$

Here K is fixed and determined by the expert based on the number of underwritings and several other important and deterministic parameters. Its value can also be considered an uncertain variable, whereas we assume that the expert fixes it as a real number for simplicity. The objective function ensures that the sponsor's return is maximized under the CAT bond's diversification in the long run. The first constraint reflects the fact that the investor cannot lose more than its principal. Therefore, considering the experts' opinion on the claims' frequency and severity, the uncertain measure of ruin (ruin index) should be kept under a pre-specified small numberε. This restriction is satisfied by the control parameter π_C. The second constraint guarantees the CAT bond to be motivating enough by respecting the investor's desire; otherwise, it is rejected by the capital market. Thus, the spread and its possible interest rate are bigger than the asymptotic expected loss. Recall that the sponsor's final gain is the difference between the premium amount and the required hedge price. Considering the solution to this optimization problem, the sponsor would decide not to purchase protection through the CAT bond when it is more expensive than the collected premium.

In uncertainty theory's framework, validating each of these inequalities would be a challenging task. In the sequel, we provide tools and approximations to reduce this difficulty and make the model more practical. This methodology is fresh and can be easily extended to pricing other instruments providing that the uncertainty theory is the basis of investigation.

Simplification of the Objective Function and Constraints

It is important to remind that working with continuous functions in uncertainty theory is more manageable than a step function. In this study, we consider the sigmoid function

$$1_A \approx \frac{1}{2}\left(1+tanh\left(\frac{x-A}{b}\right)\right). \tag{28}$$

for smoothing the indicator function. Any other similar function could be utilized with a potentially negligible disparity in the results. Observe that this function converges to 1_A as $b \to 0$. In this way, the process assures the sponsor's wealth reduces to

$$U_t^1 = u + (\pi_I - \pi_C)t - R_t + \frac{R_t}{2}\left(1+tanh\left(\frac{R_t - A}{b}\right)\right). \tag{29}$$

The following theorem presents the objective function, in the long run, using (28).

Theorem 4.1 Let R_t *be a renewal reward process with iid uncertain interarrival times* $\xi_1,\xi_2,\dots$ *and iid uncertain rewards* $\eta_1,\eta_2,\dots$ *Assume* $(\xi_1,\xi_2,\dots)$ *and* $(\eta_1,\eta_2,\dots)$ *are independent uncertain vectors with uncertainty distributions* Φ *and* Ψ*, respectively. Further, let (21) hold. Then*

$$\lim_{t\to\infty}\frac{U_t^1}{t} \to (\pi_I - \pi_C). \tag{30}$$

Proof. Notice that

$$|\tanh(\frac{x-A}{b})|\leq 1. \tag{31}$$

Using Cauchy-Schwartz inequality and the fact that $tanh\left(\frac{R_t-A}{b}\right)\to 1$as $t\to\infty$, we have

$$\left|\frac{R_t}{t}-\frac{R_t}{2t}\left(1+tanh\left(\frac{R_t-A}{b}\right)\right)\right|\leq\left|\frac{R_t}{t}\right|\left|1-\frac{\left(1+tanh\left(\frac{R_t-A}{b}\right)\right)}{2}\right|\to 0. \tag{32}$$

Consequently, the objective function becomes $\pi_I-\pi_C$.

Observe that introducing the CAT bond mostly depends on its attractiveness to the market. On the other hand, considering a CAT bond structure and the complicated covered risk, almost all CAT bonds are purchased by institutions, hedge funds, and organizations. Therefore, we emphasize that the first constraint is of higher importance. It assures that the investor's wealth never becomes negative almost sure. Let us rearrange the investor's wealth process by applying (28) and considering $b=\frac{1}{2}$ that makes this function strictly increasing. The investor's process is then redefined as

$$U_t^2=\Pi+(\Pi r+\pi_C)t-\frac{R_t}{2}\left(1+tanh\left(2\left(R_t-A\right)\right)\right). \tag{33}$$

Observe that claims' arrival times coincide with some point of the time interval, and ruin can only happen in one of these arrivals when $R_t\geq A$. Therefore, we consider $t=\sum_{k=1}^{N_t}\xi_k$, where $\xi_i=s_{i+1}-s_i$ are interarrival times and s_i's are the claims' arrival times. Consequently, the investor's risk process becomes

$$U_t^2=\Pi+(\Pi r+\pi_C)\sum_{i=1}^{N_t}\xi_i-\frac{\sum_{i=1}^{N_t}\eta_i}{2}\left(1+tanh\left(2\left(\sum_{i=1}^{N_t}\eta_i-A\right)\right)\right). \tag{34}$$

Note that the arrival time of the k-th claim is $\xi_1+\xi_2+\cdots+\xi_k$. It means that we would only be concerned about these time points because they are the only potential ruin occurring moments. Using U_k^2 indicates that one can always scale the time by considering the points where claims happen. Therefore, (34) is reduced to the following relation.

$$U_k^2=\Pi+(\Pi r+\pi_C)\sum_{i=1}^{k}\xi_i-\frac{\sum_{i=1}^{k}\eta_i}{2}\left(1+tanh\left(2\left(\sum_{i=1}^{k}\eta_i-A\right)\right)\right). \tag{35}$$

Considering the results of Vakili and Ghaffari-Hadigheh (2021), ruin for the investor happens when

$$\max_{k\geq 1}\left\{\frac{\sum_{i=1}^{k}\eta_i}{2}\left(1+tanh\left(2\left(\sum_{i=1}^{k}\eta_i-A\right)\right)\right)-(\Pi r+\pi_C)\sum_{i=1}^{k}\xi_i\right\}\geq\Pi, \tag{36}$$

for a k big enough. Let

$$g(\eta_1,\eta_2,\ldots,\eta_k)=\frac{\sum_{i=1}^{k}\eta_i}{2}\left(1+tanh\left(2\left(\sum_{i=1}^{k}\eta_i-A\right)\right)\right), \tag{37}$$

and

$$h(\xi_1,\xi_2,\ldots,\xi_k) = (\Pi r + \pi_C)\sum_{i=1}^{k}\xi_i . \tag{38}$$

Notice that $g(\eta) - h(\xi)$ is an increasing function w.r.t. η and a decreasing function w.r.t. ξ. Using Theorem 3.1, the inverse distribution of $g(\eta) - h(\xi)$ is

$$\begin{aligned} Y^{-1}(\alpha) = {} & \frac{\sum_{i=1}^{k}\Psi_i^{-1}(\alpha)}{2}\left(1 + tanh\left(2\left(\sum_{i=1}^{k}\Psi_i^{-1}(\alpha) - A\right)\right)\right) \\ & -(\Pi r + \pi_C)\sum_{i=1}^{k}\Phi_i^{-1}(1-\alpha). \end{aligned} \tag{39}$$

Further, Theorem 3.2 says that the uncertain measure of ruin is the root α of

$$\begin{aligned} \max_{k\ge 1} \{ & \frac{\sum_{i=1}^{k}\Psi_i^{-1}(1-\alpha)}{2}\left(1 + tanh\left(2\left(\sum_{i=1}^{k}\Psi_i^{-1}(1-\alpha) - A\right)\right)\right) \\ & -(\Pi r + \pi_C)\sum_{i=1}^{k}\Phi_i^{-1}(\alpha)\} = \Pi . \end{aligned} \tag{40}$$

Finally, the last constraint is reduced to the following simple form.

$$\int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)}d\alpha \le \frac{\pi_C}{(1+r)K} \tag{41}$$

It must be remarked that the ruin index, related to the investor in the constraints, can be directly calculated using equations (20) and (23). However, it results in a more complicated formula for the uncertain measure of ruin. For more details on the complexity of this approach, see Liu (2013, 2017) and references therein.

Consider an insurance company that is looking for a hedge associated with a particular risk of a catastrophic event. Moreover, assume that a market can potentially take the risk in exchange for a return significantly higher than the interest rate defined through LIBOR. The following theorem summarizes the obtained results.

Theorem 4.2. Let the insurance and investor wealth process be defined by (25) and (26), respectively. Then, an optimal solution of the optimization problem (27), if it exists, is an equilibrium price that satisfies the concerns of both the sponsor and the investor of the CAT bond.

Computational Experience

We first consider a simple illustrative example to show the performance and applicability of the proposed approach. We also examine the model on a practical instance and discuss the results against the findings using the probability theory**.**

Example 4.1 Assume that the basic risk is tied to an earthquake occurrence in a particular area for coverage of civilians property insurances. Several small insurance companies underwrite a bulk of contracts, say one million. A reinsurance company, as the sponsor, takes some risks for smaller insurance companies and wants to pool capital from the market for a hedge through a CAT bond.

To be numeric, let $r = 0.03, \pi_I = 300m, \varepsilon = 0.0001, K = 100000, \Pi = 1b, A = 2b$. To interpret this situation, we assume the severity of each claim as a lognormal uncertain variable. This uncertain variable can reasonably mimic the behavior of the situation since its mean and variance rises tremendously fast depending on the parameters (e,σ), remarking that the mean approaches to infinity for $\sigma \geq \frac{\pi}{\sqrt{3}}$. Let the severity of the claims be represented by $\mathcal{LOGN}(4,1)$ with the inverse distribution

$$\Psi^{-1}(\alpha) = exp\left(4 + \frac{\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha}\right). \tag{42}$$

We also suppose that experts' belief results in $\mathcal{L}(1,3)$ for the number of claims with the inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = 2\alpha + 1. \tag{43}$$

This assumption means that the uncertain waiting times for consecutive claims follow an uncertain linear distribution. We must make an important remark on the time frame. Notice that in empirical implementations, we do not have a continuous time-evolving. Here we assume a lattice of the time-space, determined based on the experts' opinion on the problem. In this example, the inter-arrival time distribution indicates the distance between arrival times on the lattice between 0 and 1 with different attached belief degrees. Without loss of generality, we assume the severity and frequency of the claims are independent. Therefore, the optimization problem is (numbers are scaled by $\times 10^4$)

$$\begin{aligned}
\max_{\pi_C} \quad & 30000 - \pi_C \\
s.t. \quad & \int_0^1 \frac{\left(\exp\left(4 + \frac{\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right)\right)\right)}{3-2\alpha} d\alpha \leq \frac{\pi_C}{(1+r)10} \\
& R(\alpha) \leq 0.0001 \\
& \pi_C \geq 0.
\end{aligned} \tag{44}$$

where $R(\alpha)$ is

$$\begin{aligned}
& Root\{\frac{k\left(\exp\left(4+\frac{\sqrt{3}}{\pi}\ln\left(\frac{1-\alpha}{\alpha}\right)\right)\right)}{2}(1+\tanh(2k(\exp(4+\frac{\sqrt{3}}{\pi}\ln(\frac{1-\alpha}{\alpha}))) - 200000)) \\
& -k((3000+\pi_C)(2\alpha+1)) - 100000\}.
\end{aligned} \tag{45}$$

We used the online version of the Mathematica software to solve this optimization problem. The optimal solution is 57,750,000. Suppose we designed the hedge to be covered by 1 million bonds, each having USD 1000 face value. Therefore, the spread over LIBOR for each bond is about 5.8%, meaning that if the trigger is not activated, the investor receives 8.8% interest for each unit of the CAT bond.

The next example is a comparison of results from the probability theory and uncertainty theory perspectives. The purpose is to show how important the choice of underlying indeterministic theory is and denote that those findings have different interpretations in these two different paradigms. Results indicate that probability theory and uncertainty theory are designed for problems with different natures and cannot be replaced with one another.

Example 4.2. For a real-world instance, we consider the data of Swedish fire insurance during 1958-1969 and investigate the difference of ruin index in uncertainty theory and probability theory. As detailed in Benckert and Jung (1974), denote In Thorin and Wikstad

(1977), the authors calculated the probability of ruin based on the statistical result of Benckert and Jung (1974).

We first fitted the data on some uncertainty distributions and concluded that uncertain lognormal distribution with $\mu = 1.6$ and $\sigma = 1.77$ is the best choice for the severity. Further, the linear uncertain distribution $\mathcal{L}(0,2)$ interprets the inter-arrival times quite well. Uncertain measures of ruin for $K = 100$ and $K \to \infty$, as well as for different initial capital values, were calculated. The results are compared to those obtained in Thorin and Wikstad (1977) using probability theory (see Table 1).

Table 1. Comparison of the Measure of Ruin in Probability and Uncertainty Theory Calculated for Swedish Fire Insurance's Data 1958-1969

Time	u	Probability of ruin	Uncertain measure of ruin
$T = 100$	0	0.82192	0.10238
	100	0.03701	0.10119
	1000	0.00011	0.09116
	10000	0.00001	0.03814
$T = \infty$	0	0.95238	0.10238
	100	0.55074	0.10238
	1000	0.04199	0.10238
	10000	0.00008	0.10238

Discussion

As pointed out by Liu (2017), probability theory is the best choice when sufficient data exists. It was also revealed that uncertainty theory is more conservative, meaning that it assigns higher belief degree to the occurrence of ruin. Therefore, it identifies a higher premium to cope with the risk. When time goes to infinity, the probability of ruin increases. For example, for $u = 100$, it rises from 0.03701 to 0.55074 when T goes from 100 to infinity. Further, the probability of ruin rapidly reduces by increasing the initial capital. In contrast, the uncertain measure of ruin is constant for every amount of initial capital in the long run, and does not sharply fall by increasing the initial capital for $T = 100$.

Interestingly, uncertainty theory's results seem more sensible when the results for the long run are applied. When the ultimate ruin is considered, the future does not obey the far past as the probability theory might assume, and using uncertainty theory shows some sort of independency in time. On the other hand, uncertainty theory considers the catastrophic events in the future, which is quite possible in the long run, and variation in the initial capital does not affect the risk measure. We also observed that when the initial capital rises significantly, e.g., to $u = 10^7$ and higher, the uncertain measure of ruin drops remarkably to almost zero.

In defining a CAT bond, as it is visible from Table 1, satisfying the ruin-related constraints requires a higher price for the CAT bond. There might be some reasons behind this conclusion. First, the data is available only for a short period between the possible past and future time span. Secondly, the fast growth of populations and advances in the cities' development is not treated in the modeling process. Observe that limited data or untrustworthy information is the main motivation for referring to an expert and using uncertainty theory. Moreover, the results provided by probability theory are useful in a short period after they are obtained. Therefore, the risk of ruin is less than the probability theory, as reflected in Table 1.

Conclusions and Remarks

In this study, the expert opinion was considered as the main tool in problem modeling in an uncertain environment. It was demonstrated that uncertain optimization could be considered as a practically implementable paradigm for pricing securities, especially insurance-linked ones such as CAT bonds. A highly complicated inequality, the ruin-related constraint, was approximated by a reliable and tractable substitute inequality. Nonetheless, we considered the optimization with only a single objective; the multi-objective version would be more interesting and realistic. Moreover, modeling CAT bond's structure and its pricing problem, with re-insurance policy, will be a more sensible framework, too. This problem can also be considered with multiple objectives where each party's favor is an objective.

Another approach would be to utilize the expected value of the involved uncertain variables in the model and handle the optimization problem using vector optimization as a deterministic and powerful tool in mathematical programming. Optimally fitting collected data from experts' opinions into an uncertainty distribution and measuring its parameters is another research line in practice.

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