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Shortest path Ranking Model With the Ability to Change the Importance of Inputs and Outputs: Case Studies from Education

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ABSTRACT

This paper presents a novel super efficiency model based on the Andersen-Petersen model, which serves as a bridge between input-oriented and output-oriented models. The proposed model defines a path from the deleted decision-making unit to the efficiency frontier with the shortest step length. Initially formulated as a nonlinear programming model, the developed model is transformed into a multi-objective linear programming model and then further simplified into a linear programming model through variable changes and the application of nonlinear programming solving methods. The feasibility of the proposed path is discussed, and a weighted version of the shortest path model is introduced to incorporate preferences regarding the relative importance of inputs or outputs. Addressing a weakness of the AP methods, the inability to prioritize weights is resolved in the developed model. Real case studies in the context of Iranian education are conducted, and the results of the AP and shortest path analyses are analyzed to validate the proposed method.

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1. Introduction

Regarding the presence of multiple efficient units, it is possible for multiple units to have an efficiency value of 1 simultaneously. In such cases, the issue of ranking becomes important. Andersen and Petersen (1993) proposed a method called "super-efficiency" (AP) to address this. In this method, the Decision Making Unit (DMU) under evaluation is removed from the Production Possibility Set (PPS), and the remaining units utilize the Data Envelopment Analysis (DEA) model.

The AP model continues to be one of the most widely used ranking methods, but various studies have raised certain issues over time. One such issue is the infeasibility of some DMUs, which has been addressed through the introduction of new models and modifications to the AP model. Banker & Gifford (1988, p. 88) proposed the BG model, which resolves the infeasibility challenge by adding a coefficient to the objective function of the evaluated unit. They demonstrated that a feasible solution for this coefficient always exists. Banker and Chang (2006) used simulation to show that the efficiency performance measured by the BG model is superior to that of the AP model. Koeisova and Paleckova (2017) employed a non-radial and non-oriented super-efficiency SBM model, assuming variable returns to scale, to analyze the performance of twenty-two domestic commercial banks in the Czech Republic and Slovakia. Amirteimoori et al. (2017) investigated the infeasibility of the input-based super-efficiency model and proposed a solution by adding two virtual DMUs with mean values. Lin & Li (2020) introduced two diversification super-efficiency models for discriminating the performance of efficient mutual funds in the financial market. Gerami et al. (2021) presented a model that calculates efficiency and super-efficiency scores based on the corresponding slack values for both input/output and output/input ratios, depending on the orientation of the production frontier.

The MAJ (Mehrabian, Alirezaee, Jahanshahloo 1999) model follows a series of steps, where the JHF (Jahanshahloo, Hosseinzadehlotfi, F. Shoja) model is initially presented. The JHF model enables non-radial movement towards the PPS by separately increasing input and decreasing output. The objective function is defined as the minimum sum of distance coefficients of inputs and outputs required to reach the frontier. This model is obtained by modifying certain variables in the MAJ model to make it input- and output-oriented, thereby reducing the size of the objective function by fixing some weights. Feasibility issues have been demonstrated in all the aforementioned models.

In addition to addressing the infeasibility issue, the AP method can tackle other concerns. One such concern, introduced by Banker et al. (2017), is the lack of ranking popularity. This means that a unit's expectations for its performance significantly differ from the ranking results. We argue that the AP method unconsciously prioritizes based on the positioning of inputs and outputs, regardless of whether the model is input-oriented or output-oriented. This claim is made in this paper because the ranks change as the path changes. In certain cases, a conscious prioritization of outputs and inputs may be required for ranking, necessitating separate weighting of inputs and outputs relative to each other. Another weakness of the AP method is its lack of ranking strategy when some DMUs have similar rankings. Banker et al. (2017) found that ultra-efficiency-based methods effectively identify outliers but not the classification of efficient units.

To highlight the importance of prioritization, let's consider two lubrication systems, A and B, both containing 100 kg of soybeans. Device A has 16 liters of oil, 80 kg of soybean meal, and 10 kg of other materials, while device B has 30 liters of oil, 50 kg of soybean meal, and 10 kg of other materials. The AP model may rank A higher than B, even though the main task of these devices is to extract oil rather than produce soybean meal. The AP method lacks the ability to prioritize inputs and outputs.

In addition to the issues of infeasibility and high-efficiency scores of BG compared to the AP model, determining a specific path leads to unintentional prioritization of inputs and outputs, making it difficult to apply weighting. Weighting is typically applied to the objective function, which consists of multiple components and extends beyond the zero-one distance in non-super-efficiency, thereby deviating from the concept of efficiency. While the modified JAM method addresses this weakness, it no longer supports weighting and prioritization. Another challenge is that this non-radial model does not follow a specific model to evaluate efficiency and create super-efficiency.

Amirteimoori et al. (2017) conducted a study that addressed the difficulty of infeasibility by introducing additional DMUs, which prevent infeasibility by expanding the frontier.

Weight restriction methods involve applying weights within a specific range and creating a new model, which in turn alters the efficient frontier and affects ranking. The most notable weight restriction models are AR (Assurance Region) and the Con ratio method. However, these methods have limitations: weight restrictions are confined to a specific range, and direct weighting is not feasible. Gkouvitsos & Giannikos (2021) proposed a model that combines the AP method with multi-criteria and weight restriction methods.

Evaluation methods can be categorized as parametric and non-parametric. Data envelopment analysis belongs to the parametric category, and studies have often been conducted to compare parametric and non-parametric methods. One of the popular parametric ranking methods is TOPSIS, where positive and negative ideals are defined, and rankings are based on the distance from these ideals. Ersoy (2021) conducted a comparison between super-efficiency methods in DEA and TOPSIS.

Thus far, there have been no studies on the change in frontier motion direction in BCC (Banker, Charnes, Cooper) models and other radial models based on nonlinear problems. Therefore, this study is novel. While some research has focused on resolving infeasibility in super-efficiency methods, limited studies have explored the direct application of weights to these models. The most prominent studies in this area are summarized in the table below.

The novelty of this study lies in the model's ability to determine the path to the frontier, solve the infeasibility challenge, and incorporate weighting.

Table 1. Comparison between the shortest path model and other models

Model	Properties	Model Defects	Shortest path model benefits
BG (Banker, Gifford)	Resolve the infeasibility of super-efficiency	Inability to weighting	Weighting capability
Assurance Region Con ratio method	Restriction of multiplier	Weight control is performed in a certain range. It is not possible to apply weight directly	applying direct weight to the model
JHF (Jahanshahloo,Hosseinzadeh lotfi, F. Shoja)	resolve the infeasibility ability to Weighting	There is no specific base model from which the super-efficient model is derived. The weight is applied to the objective function, and the initial model is not a model for calculating efficiency, and the value of objective function does not correspond to the definition of efficiency (between zero and one).	It is derived from the BCC model by changing the direction. Weight are located in constraints and the value of the objective function is between zero and one based on $\theta + 1$
MAJ (modified JHF, (Mehrabian, Alirezaee, Jahanshahloo)	JHF objective function problems were removed	Inability to weighting	Weighting capability

Various models have attempted to address the challenges of the AP method, including infeasibility and the lack of weighting ability. Table 1 compares the characteristics and drawbacks of the most well-known proposed models with the shortest path model.

The objective of this study is to develop a highly efficient technique based on a nonlinear programming model that follows the ideal path determined by the solution of the shortest path model. This leads to a nonlinear model that has been linearized using Nonlinear Programming (NLP) methods. One notable feature of the proposed method is that the direction of movement towards the Production Possibility Set (PPS) is determined by the model itself, without imposing a predefined path. The path is configured to reach the PPS with the shortest step length. The closest path from the evaluated DMU to the PPS is determined by the input and output distance vectors. However, in the AP method, where the path is predetermined, certain inputs or outputs may have a greater or lesser impact on the ranking, depending on whether it is input-oriented or output-oriented. This discrepancy might explain the lack of popularity of some ranking results. In this context, "unpopularity" refers to the fact that the model's ranking outcomes do not align with the expectations of DMUs in the real world.

The main motivation behind this study is to eliminate this confusion and provide a standardized path determined by the model itself, while also addressing the differences in ranking results between AP's input-oriented and output-oriented methods, as well as the variation in ranking results due to path changes. Another motivation for developing this model is that, within the same objective function values, there are alternative solutions classified differently by the AP method. Additionally, this study addresses the infeasibility issue of the AP method through the proposed approach. The model itself determines the path based on the size of the inputs and outputs, without imposing any specific input and output semantic prerequisites. The ability to apply priorities based on the relative importance of inputs and outputs is a significant contribution of the proposed model.

The remaining sections of this white paper are organized as follows: Section 2 provides an overview of the basic DEA model background. Section 3 introduces the shortest path model. In Section 4, the NLP model is transformed into a Multi Objective Linear Program (MOLP) and then to LP. Section 5 presents a weighted shortest path model that prioritizes the relative importance of inputs and outputs. Section 6 includes a practical case study analyzing Iran's 32 states in the primary education system. Finally, Section 7 concludes with final remarks and discusses future research areas.

2. Background

Suppose we have n DMUs, each using m inputs and producing s outputs. The inputs and outputs of all DMUs are assumed to be strictly positive. The BCC input-oriented model of Banker et al. (1984) to assess the relative efficiency of DMU0 is as follows:

$$\begin{aligned} &\theta_{BCC} = Min\,\alpha \\ &s.t. \sum_{j=1}^{n} \lambda_j x_j \le \alpha x_0 \\ &\sum_{j=1}^{n} \lambda_j y_j \ge y_0 \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \ge 0,\ j = 1,\ldots,n \end{aligned} \tag{1}$$

If you have multiple efficient DMUs, you need to rank them. One of the most popular ranking methods is AP model. This method revalues the DMUs by eliminating the corresponding efficient ones. The corresponding PPS after eliminating DMU0 is:

$$P \setminus (x_0, y_0) = \left\{ \begin{array}{l} (x, y) \mid x \ge \sum_{j=1}^{n} \lambda_j x_j,\ y \le \sum_{j=1}^{n} \lambda_j y_j, \\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0,\ j = 1,\ldots,n,\ j \ne 0 \end{array} \right\} \tag{2}$$

This PPS defines the feasible set of the AP ranking model as follows:

$$\begin{aligned} &\theta_{BCC} = Min\,\alpha \\ &s.t. \sum_{j=1}^{n} \lambda_j x_j \le \alpha x_0 \\ &\sum_{j=1}^{n} \lambda_j y_j \ge y_0 \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \ge 0,\ j = 1,\ldots,n,\ j \ne 0 \end{aligned} \tag{3}$$

Thus, instead of the basic BCC model, other measures such as slack-based measures are used in the ranking process. The slacks-based model produces single-value efficiency scores. Kao and Liu (2020) applied the cross-efficiency idea with this measure to a real case study involving the selection of the most efficient robot for production. The results help identify the top-ranked robot.

3. Shortest path model

This section runs a different path than AP method. In fact, this method is a combination of input-oriented and output-oriented AP methods, both of which are considered at the same time. We are only investigating Variable Returns to Scale (VRS) technology. Otherwise, it will be pointed out.

Let (x_0, y_0) be an extreme efficient DMU. By omitting DMU0, the PPS is as (2). Now we move from (x_0, y_0) in the direction of $(d_x, -d_y)$ by a step length of θ toward $P \setminus (x_0, y_0)$. We would have $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \theta \begin{pmatrix} d_x \\ -d_y \end{pmatrix} \in P \setminus (x_0, y_0)$. Therefore, the following nonlinear model is obtained:

$$\theta_{BCC} = Min\theta$$
$$s.t.\sum_{j=1}^{n}\lambda_j x_j \le x_0 + \theta d_x$$
$$\sum_{j=1}^{n}\lambda_j y_j \ge y_0 - \theta d_y \quad (4)$$
$$\sum_{j=1}^{n}\lambda_j = 1$$
$$\lambda_j \ge 0, j = 1,\dots,n, j \ne 0$$

The desired path is illustrated in figure 1 by $\overrightarrow{BP}$.

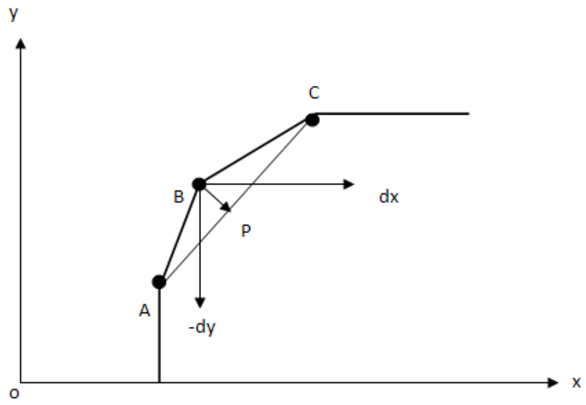

Figure 1. The shortest path

If we define $d_y = 0$ and $d_x = x_0$, then we have the AP input-oriented method, and if $d_y = y_0$ and $d_x = 0$, the model is AP output-oriented. The step length obtained from model (4) is smaller than the AP input-oriented model. In fact, in the AP method, we impose a predefined path on the model. In one-dimensional case, the AP input-oriented and output-oriented cases, matches with d_x and d_y in model (4), respectively, and this does not hold for higher dimensions.

Theorem 1. Suppose that DMU0 is extreme efficient and θ^* is the optimal solution of model (4) and α^* is the optimal solution of AP input-oriented model (3), then $1+\theta^* \le \alpha^*$.

Proof: Let $\left(\lambda_1^*,\lambda_2^*,\dots,\lambda_n^*,\alpha^*\right)$ be an optimal solution of model (3), then $\left(\lambda_1^*,\lambda_2^*,\dots,\lambda_n^*,\alpha^*-1,x_0,0\right)$ is a feasible solution of model (4), where $\lambda_j \ne \lambda_0$, $j = 1,\dots,n$. Therefore, $\theta^* \le \alpha^* - 1$ or $1+\theta^* \le \alpha^*$.

Similarly, for any other predetermined path, the objective function value of model (4) is smaller than the corresponding related model. So, we call this model as "shortest path model".

Based on the objective value of model (4), all DMUs can be classified with their corresponding θ as follows:

1. If $\theta > 0$, that DMU is extreme efficient.
2. If $\theta = 0$, that DMU is non-extreme efficient.
3. If $\theta < 0$, that DMU is inefficient.

Clearly, the discussed categories in ranking context are (1) and (2), and the category (3) is not the subject of this study. Based on the value of θ, this model has the ability to rank extreme efficient units.

One of the AP difficulties, which rarely happens, is that the objective function value of two or more DMUs may be equal while in model (4), the alternative ranking of $\|d\|_2$ or $\|d\|_\infty$ could be used.

4. Linearization of the proposed model

The nonlinear problem (4) has three basic parameters, one of them is constant, and other two are variable. The constant parameter is the starting point that is the omitted DMU from PPS denoted by (x_0, y_0). The next parameter is the direction of movement, along which the objective function decreases uniformly (for minimization problem) that is $d_x = (d_1,\ldots,d_m)$ and $d_y = (d_1,\ldots,d_s)$ corresponding to inputs and outputs, respectively. The third parameter is the step length of the movement, which is θ in the objective function of model (4). One can use various norms of the vector $\vec{d}$ that is the direction to move to the frontier. For example, Jahanshahloo et al. (2004) used l_1-norm.

Model (4) can be solved using nonlinear algorithms, and both d_x and d_y could be obtained. This can be done using different nonlinear methods, in particular the cutting plan method, which is a method for solving nonlinear problems close to linear problems. But, we want to convert the model to linear models to get more accurate solutions. Hence, using the idea of Kuosmanen (2005) and MONLP problems, the model will be linearized as follow:

$$
\begin{aligned}
& Min\left(\theta_1,\theta_2,\ldots,\theta_m,\rho_1,\rho_2,\ldots,\rho_s\right) \\
& s.t. \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j x_{ij} \le x_{i0} + \theta_i d_{ix}, i = 1,\ldots,m \\
& \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j y_{rj} \ge y_{r0} - \rho_r d_{ry}, r = 1,\ldots,s \\
& \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0,\ j = 1,\ldots,n,\ j \ne 0
\end{aligned} \tag{5}
$$

At the first stage, model (5) is converted into a MOLP as follows:

We apply the variable transformation of

$$d_{ix} = \frac{\alpha_i}{\alpha_i + \mu_i}, \qquad \theta_i = \alpha_i + \mu_i$$

So we have $\mu_i = \left(1 - d_{ix}\right)\theta_i$, $\alpha_i = \theta_i d_{ix}$. Similarly, for outputs we apply

$$d_{ry} = \frac{\omega_r}{\omega_r + \xi_r}, \qquad \rho_r = \omega_r + \xi_r$$

Then we have $\mu_i = \left(1 - d_{ry}\right)\rho_r$, $\omega_r = \rho_r d_{ry}$.

The problem is turned into MOLP after replacing new defined variables. The min max method for solving the problem is:

$$
\begin{aligned}
& MinMax\left(\theta_1,\theta_2,\ldots,\theta_m,\rho_1,\rho_2,\ldots,\rho_s\right) \\
& s.t. \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j x_{ij} \le x_{i0} + \theta_i - \mu_i, i = 1,\ldots,m \\
& \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j y_{rj} \ge y_{r0} - \rho_r + \xi_r, r = 1,\ldots,s \\
& \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0,\ j = 1,\ldots,n,\ j \ne 0
\end{aligned} \tag{6}
$$

and the final problem is:

$$
\begin{aligned}
& Min\theta \\
& s.t. \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j x_{ij} \le x_{i0} + \theta_i - \mu_i, i = 1,\ldots,m \\
& \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j y_{rj} \ge y_{r0} - \rho_r + \xi_r, r = 1,\ldots,s \\
& \sum\nolimits_{j=1,\, j\neq 0}^{n} \lambda_j = 1 \\
& \theta \ge \theta_i, i = 1,\ldots,m \\
& \theta \ge \rho_r, r = 1,\ldots,s \\
& \lambda_j \ge 0,\ j = 1,\ldots,n,\ j \ne 0
\end{aligned} \tag{7}
$$

The initial direction can be calculated by variable variation.

The reason why AP model is infeasible in some cases is that by deleting a DMU, the new frontier may not be in the direction of movement, but in model (4) and consequently, model (7), the frontier is always crossed. This is because the movement direction toward the frontier is always diagonal. So, the model is ever feasible.

Theorem 2. Model (7) is always feasible.

Proof: We can define a feasible solution to model (7) as shown below:

$$\lambda_j = \begin{cases} 1 & j = 1,\dots,n,\ j \neq 0 \\ 0 & j = 0 \end{cases}, \quad \theta_i = \sum_{j=1, j\neq 0}^{n} x_{ij}, \quad \mu_i = 0, i = 1,\dots,m$$

and we set $\rho_r = y_{r0} - \sum_{j=1, j\neq 0}^{n} y_{rj}$ and $\xi_r = 0$.

5. Prioritization of inputs, outputs and weighting

In some cases, the ranking process may need to prioritize between the relative importance of inputs and outputs. In this situation, it seems that they need to be weighted accordingly. Model (5) does this by simply altering the route in the manner described below, without explicitly assigning weights to the goal function. The significance of inputs and outputs is often depicted by moving away from or toward the input and output vectors. Suppose we want to impose the weight w_1 to the first input of the model (5). We do this as follows:

$$\sum_{j=1, j\neq 0}^{n} \lambda_j x_{1j} \leq x_{10} + \frac{\theta_1 d_{1x}}{w_1}$$

Now if we similarly impose w_2 to the first input where $w_2 > w_1$ we have:

$$\sum_{j=1, j\neq 0}^{n} \lambda_j x_{1j} \leq x_{10} + \frac{\theta_1 d_{1x}}{w_2}$$

We want to examine the effect of w_1 and w_2 on the objective function to see if the objective function achieves a larger value?

Assume the optimal value of the input in model (5) is as $\theta^* d^*_{x_1} = k_1$. When we impose w_1 or w_2 on the model, the nonlinear model will try to reach the same optimal point; the larger the denominator, the larger the numerator to reach k_1 value.

Considering that $w_2 > w_1$, then $\theta^*_2 d^*_{x_2} > \theta^*_1 d^*_{x_1}$ where $\theta^*_1, d^*_{x_1}$ are the optimum values for weight w_1 and $\theta^*_2, d^*_{x_2}$ are the optimal values for weight w_2. Now, we have the following cases for comparing θ^*_1, θ^*_2 and $d^*_{x_1}$, $d^*_{x_2}$ with each other.

- Case 1. $\theta^*_2 < \theta^*_1$ and $d^*_{x_2} < d^*_{x_1}$. This case does not occur, because in this case, the condition $\theta^*_2 d^*_{x_2} > \theta^*_1 d^*_{x_1}$ is not met.
- Case 2. $\theta^*_2 < \theta^*_1$ and $d^*_{x_2} > d^*_{x_1}$. This case does not occur, because if $d^*_{x_2} > d^*_{x_1}$, due to the path change and the new path is far from the frontier, so surely $\theta^*_2 > \theta^*_1$. One should note that in this case, the remaining inputs and outputs have no weights, and the only purpose is examining the effect of the large weights on the first input.
- Case 3. $\theta^*_2 > \theta^*_1$ and $d^*_{x_2} < d^*_{x_1}$. This case does not happen either because the longer step length is due to a distance greater than the frontier, not less.
- Case 4. $\theta^*_2 > \theta^*_1$ and $d^*_{x_2} > d^*_{x_1}$. This is true and it means that the larger the weight of denominator, the larger the effect on the next objective function. This is also true for outputs.

Above discussion about extreme efficiency leads to positive d_x and θ.

One of the advantages of this weighting method is that there is no difference in weighting the inputs ore outputs, meaning that it is possible to prioritize inputs and outputs separately or relative to each other.

Suppose that the weight of input i is set to w_i. The nonlinear weighted model is written as model (8) below:

$$
\begin{aligned}
& Min\theta \\
& \text{subject to} \\
& \sum_{j=1, j\neq 0}^{n} \lambda_j x_{ij} \leq x_{i0} + \frac{\theta d_{ix}}{w_i}, i=1,\dots,m \\
& \sum_{j=1, j\neq 0}^{n} \lambda_j y_{rj} \geq y_{r0} - \frac{\theta d_{ry}}{w_r}, r=1,\dots,s \\
& \sum_{j=1, j\neq 0}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0, j=1,\dots,n, j\neq 0
\end{aligned} \tag{8}
$$

Now with the variable variation of $\frac{d_{ix}}{w_i} = \frac{\alpha_i}{\alpha_i + \mu_i}$, $\theta_i = \alpha_i + \mu_i$, we get $\theta_i d_{ix} = w_i \alpha_i$ or $\theta_i d_{ix} = w_i \theta_i - w_i \mu_i$. Applying these new variables, the following linear model is obtained:

$$
\begin{aligned}
& Min\theta \\
& \text{subject to} \\
& \sum_{j=1, j\neq 0}^{n} \lambda_j x_{ij} \leq x_{i0} + w_i \theta_i - w_i \mu_i, i=1,\dots,m \\
& \sum_{j=1, j\neq 0}^{n} \lambda_j y_{rj} \geq y_{r0} - w_r \rho_r + w_r \xi_r, r=1,\dots,s \\
& \sum_{j=1, j\neq 0}^{n} \lambda_j = 1 \\
& \theta \geq \theta_i, \theta \geq \rho_r \\
& \lambda_j \geq 0, j=1,\dots,n, j\neq 0
\end{aligned} \tag{9}
$$

Where $\sum_{i=1}^{n} w_i = 1$, $\sum_{r=1}^{s} w_r = 1$, $w_i > 0$, and $w_r > 0$. Of course, the method for output weights is similar.

As you can see, the weighting was accomplished by altering the route, which altered the step length and ranking. This means that if you've previously chosen a route for your model, you're unintentionally prioritizing inputs and outputs. This is accomplished using established high-efficiency models, such as the AP technique.

6. Case Study from Education

This section gives two example of education.

Example 1: In this example, a real case study from primary education of 32 provinces of Iran is conducted to survey relative performance of them with respect to ranking context. Each province is considered as a distinct DMU. After consulting with primary education experts, the inputs and outputs are considered as Table 2.

Descriptive statistics of inputs and outputs in 2018 are shown in Table 3.

After running BCC model, we found 16 efficient DMUs. DEA efficiency ranking results for the AP input-oriented method, the AP output-oriented method, the shortest path method, and weighted shortest path method are illustrated in Table 4.

Table 2. Inputs and outputs description

inputs
number of educational buildings available for elementary level
number of sixth grade teachers, which is the last grade in elementary level
number of students in the sixth grade
Outputs
number of students have been accepted at the sixth grade and started seventh grade without leaving their education
number of sixth grade students who have been accepted at the sixth grade and they might leave or not
Number of 11-year-old students of sixth grade, i.e. students who had come to the sixth grade in the actual age of sixth grade without effect son their entry and without repeating a grade

Table 3. Descriptive statistics of inputs and outputs

	Min	**Max**	**Mean**	**STD.**
Input 1	402	4494	1795.344	1097.738
Input 2	344	3721	1268	907.7294
Input 3	8104	99589	34964.53	25101.29
Output 1	7915	94082	33570.88	24074.84
Output 2	8091	98973	34819.44	24975.26
Output 3	7513	85878	30977.63	21899.2

Table 4. results for different ranking methods

efficient units	**rank by AP input-oriented**	**rank by AP output-oriented**	**rank by shortest path method**	**rank by weighted shortest path method**
DMU2	8	7	7	7
DMU4	14	12	15	15
DMU5	3	1	6	6
DMU6	4	infeasible	3	3
DMU8	1	2	2	1
DMU9	10	8	8	8
DMU12	infeasible	5	1	2
DMU15	7	3	9	10
DMU16	2	infeasible	5	4
DMU20	5	infeasible	12	11
DMU21	12	10	11	12
DMU23	6	4	10	9
DMU24	9	6	14	13
DMU28	11	9	4	5
DMU29	15	13	16	16
DMU31	13	11	13	14

As observed, the ranking results can differ for AP's input-oriented and output-oriented approaches. For instance, DMU12 in the first column and DMU6, DMU16, and DMU20 in the second column are infeasible. However, DMU12 is ranked 1 and 2 in the shortest path and weighted shortest path methods, respectively. Hence, when the AP method is infeasible for certain DMUs, alternative methods can be utilized. Additionally, the impact of the chosen path is evident from Table 4.

In the weighted shortest path method, the decision maker assigns importance and priority to inputs and outputs through the following weights: [weights for inputs], and [weights for outputs].

A thorough sensitivity analysis is required to examine the influence of inputs and outputs and their distances from the ranking path. This topic can be extensively studied. Similarly, the issue of popularity can be explored through simulation in numerous studies. However, we have summarized the findings regarding these two topics. One intriguing approach to sensitivity analysis involves perturbing data simultaneously in all DMUs, as demonstrated by Khodabakhshi et al. (2014) in the context of super-efficiency measures when input relaxation is allowed.

It is already known that DMU28 exhibits satisfactory educational conditions, and the AP ranking of 11 contradicts the province's popularity. A rank of 4 from the shortest path method aligns better with the province's popularity. Upon examining the data values of this DMU, it becomes apparent that its third input is relatively large, which indicates a weakness in measuring efficiency performance. It seems that the AP method has assigned a significant weight to the third input. Supporting this claim, in the weighted shortest path, when we increase the weight of the third input (0.5) compared to the other inputs (0.2 and 0.3), we are essentially amplifying the weakness, resulting in a lower ranking, dropping from 4th to 5th. Similar discussions can be made regarding DMU20. In the AP input-oriented ranking, DMU20 is placed higher than expected based on its low popularity. However, the shortest path method provides a more realistic rank by assigning a low weight to the first output and a high weight to the second output, considering the low value of the first output in DMU20.

In terms of comparison with the methods in Table 1 and other previous studies, it is necessary to clarify that most of the models are employed to address the infeasibility of the AP method. Therefore, the Shortest Path model may not possess a distinct advantage over these models. The Shortest Path model has the advantage of direct weighting in addition to feasibility. Among the models, the JHF model is the only one capable of direct weighting while also being feasible. However, as indicated in

Table 1, this model, due to its objective function value, does not correspond to the definition of efficiency, leading to difficulties in direct weighting. Consequently, a modified model called MAJ (Modified JHF) was created to resolve the objective function value issue and ensure feasibility, but it no longer retains the ability to weigh.

Based on the information provided, we will employ the TOPSIS method, a non-parametric method, to compare the weighting capability of the Shortest Path model in the next example.

Example 2: With a focus on the education case study, the second example pertains to education and involves three inputs and three outputs. The first three inputs are as follows: student per capita (budget to student ratio), per capita student space (ratio of educational space to student), and student to teacher ratio. The three outputs consist of net enrollment rate (ratio of student population to typical age of the course), promotion percentage (proportion of students advancing to a higher grade), and passing percentage.

In this example, the data ranking is initially presented using the input-oriented AP model. Subsequently, the data ranking is provided using the shortest path method and the weighted shortest path method. Finally, for simulation and comparison with parametric methods, the shortest path is compared to the TOPSIS method, both with and without weights, similar to the weights employed in the shortest path method.

Table 5. Input output data

DMUs	**inputs**			**outputs**		
	student per capita	**Per capita student space**	**teachers to students**	**net enrollment**	**promotion**	**passing**
province1	21.22	3.54	0.0442	98.99	98.45	99.15
province2	18.68	2.86	0.0395	97.56	97.96	99.48
province3	26.28	4.04	0.0517	98.83	98.93	99.61
province4	21.44	3.81	0.0387	98.84	98.22	99.07
province5	16.28	1.95	0.0294	98.81	98.70	99.60
province6	35.13	5.16	0.0664	98.46	98.58	99.35
province7	22.81	3.33	0.0397	98.89	97.41	98.26
province8	16.79	2.39	0.0350	98.00	98.48	99.39
province9	15.13	1.78	0.0275	98.00	98.14	99.29
province10	28.11	4.28	0.0509	98.65	98.23	98.88
province11	29.09	8.02	0.0588	98.71	97.44	98.40
province12	22.00	2.71	0.0425	97.93	96.82	98.23
province13	26.61	3.80	0.0515	98.25	96.65	97.87
province14	20.45	3.21	0.0397	97.42	97.21	98.55
province15	25.40	3.59	0.0481	99.15	98.73	99.37
province16	23.70	3.60	0.0403	98.93	98.08	98.97
province17	17.40	2.33	0.0387	91.31	89.71	93.39
province18	23.26	3.67	0.0437	98.49	97.48	98.39
province19	21.06	3.73	0.0397	99.08	98.63	99.26
province20	18.69	3.03	0.0293	98.56	97.72	98.76
province21	26.75	3.50	0.0528	98.64	98.47	99.49
province22	25.66	3.49	0.0473	96.94	96.78	98.13
province23	24.17	3.62	0.0509	98.11	98.12	99.02
province24	32.86	5.19	0.0628	98.37	96.72	97.76
province25	20.95	3.44	0.0382	98.17	97.30	98.66
province26	24.58	2.98	0.0506	98.91	98.92	99.39
province27	26.57	3.34	0.0479	97.95	97.33	98.54
province28	26.44	4.16	0.0486	99.07	99.26	99.62
province29	19.99	4.14	0.0332	99.10	98.41	99.14
province30	21.85	3.63	0.0431	97.72	97.04	98.11
province31	24.95	3.77	0.0475	98.79	98.58	99.28
province32	22.55	5.79	0.0357	99.28	98.21	99.01
weight	0.50	0.30	0.2000	0.50	0.30	0.2000

Table 6. Results for different ranking methods

DMUs	input-oriented AP		shortest path		weighted shortest path		TOPSIS	
	θ	Rank	θ+1	Rank	θ+1	Rank	Rank of non-weighted	Rank of weighted
province1	0.9	10	0.9877	18	0.9755	19	12	10
province2	0.85	12	0.9892	15	0.9785	16	6	6
province3	0.81	13	0.9935	10	0.9871	11	27	26
province4	0.77	15	0.9913	11	0.9825	12	13	14
province5	2.03	1	1.4807	2	3.4036	2	2	2
province6	0.45	30	0.9627	32	0.9255	32	31	31
province7	0.76	16	0.9909	12	0.9817	14	10	12
province8	0.94	7	0.9937	9	0.9873	10	3	3
province9	1.09	3	2.15	1	6.75	1	1	1
province10	0.57	25	0.9782	27	0.9564	26	28	29
province11	0.55	28	0.9705	30	0.941	30	32	32
province12	0.68	20	0.985	20	0.997	9	g	8
province13	0.58	23	0.98	25	0.954	27	25	25
province14	0.73	17	0.988	17	0.976	18	8	7
province15	1.3	2	1.0817	5	1.408	6	21	21
province16	0.78	14	0.9908	13	0.982	13	16	17
province17	0.86	11	0.9888	16	0.978	17	4	4
province18	0.68	21	0.985	21	0.97	21	18	18
province19	1.031	4	1.0189	6	1.947	4	11	11
province20	0.98	6	0.9995	8	0.9991	8	5	5
province21	0.5	29	0.9763	29	0.9526	29	24	24
province22	0.58	24	0.9802	24	0.9604	24	20	20
province23	0.73	18	0.977	28	0.9538	28	22	19
province24	0.94	8	0.9657	31	0.9313	31	30	30
province25	0.57	26	0.9898	14	0.9795	15	9	9
province26	0.94	9	0.987	19	0.9739	20	17	16
province27	0.57	27	0.98	26	0.96	25	19	23
province28	infeasible	-	1.33	3	2.65	3	26	27
province29	1.02	5	1.016	7	1.0824	7	14	13
province30	0.69	19	0.985	22	0.9688	22	15	15
province31	0.65	22	0.982	23	0.964	23	23	22
province32	infeasible	-	1.1549	4	1.7	5	29	28
weight	0.50	0.30	0.20	0.50	0.30	0.20		

The shortest path model, with an efficiency value of 1 + θ, is capable of ranking both efficient and inefficient units. To compare the shortest path method with the input-oriented AP method, as well as with the parametric TOPSIS method, all units have been ranked.

Out of the 32 units, 6 units are efficient. Among these units, unit number 9 is ranked first in the shortest path method and in the TOPSIS method without weights, but it is ranked third in the AP method. On the other hand, unit number 5 exhibits a different pattern, as it is ranked 2nd in the shortest path method and TOPSIS, but holds the 1st rank in the AP method. Generally, the ranking of efficient units is closer between the shortest path method and TOPSIS compared to the AP method. Considering the TOPSIS rank as a criterion for comparing the AP method with the shortest path method, the AP method has an average ranking distance of 6.33 across all units (efficient and inefficient), while the shortest path method has an average distance of 5.13 from the TOPSIS ranks.

This example demonstrates that the shortest path method performs better than the AP method when compared to a parametric method. Furthermore, the infeasible units in the AP method are excluded in all methods.

Regarding the weighted model of the shortest path and its comparison with the TOPSIS method, the weights applied in both methods are exactly the same. These weights are: first input 0.5, second input 0.3, third input 0.2, first output 0.5, second output 0.3, and third output 0.2.

Analyzing the impact of these weights on comparing the weighted shortest path method with the non-weighted shortest path method, we find that the rankings for the first, second, and third positions remain unchanged in the weighted state. However, there are changes in the subsequent ranks. For instance, the 4th rank in the non-weighted shortest path method becomes the 5th rank in the weighted model, the 5th rank becomes the 6th rank, and the 6th rank becomes the 4th rank in the weighted model.

In the case of TOPSIS, the changes resulting from weight application are minimal, with the first six ranks remaining the same in both the weighted and non-weighted modes. This highlights the greater influence of weights in the shortest path method compared to the TOPSIS method.

7. Conclusion and future works

Due to the differences in ranking results in the AP method and the presence of infeasible units, a new ranking method called the shortest path has been introduced based on super-efficiency models, specifically the AP method. This model determines a unique path to the frontier, regardless of whether it is input-oriented or output-oriented, making it a standardized path. The proposed model maintains the nature of the super-efficiency method and serves as a step towards the development of the AP model. The weighting argument was introduced to demonstrate the effect of the path on ranking, aiming to determine priorities for inputs and outputs. However, the unweighted model of the shortest path does not prioritize inputs and outputs, resulting in a ranking that is more consistent with reality.

In an example, the differences between input-oriented and output-oriented ranking in the AP method, as well as the presence of infeasible units, were observed. The feasibility of the shortest path model was proven and demonstrated through numerical examples.

Another notable feature of this model is its ability to incorporate weights on inputs and outputs without altering the nature of the model or the efficient frontier. Weighting is applied in constraints and can be observed in the efficiency value and unit ranking. This effect was demonstrated both theoretically and with a numerical example.

Comparing the shortest path method with a parametric method revealed that the ranks in the shortest path method were closer to the ranks of the parametric method compared to the AP method. The impact of weights on rank displacement was greater in the shortest path method than in the parametric method.

In summary, the shortest path model offers four advantages:

1. The uniqueness of the path and the model's ability to choose the path, preventing multiple paths and different rankings.
2. Feasibility of the model in all cases, achieved through the approach towards the efficient frontier.
3. Easy capability for weighting inputs and outputs, where weights are applied in constraints, resulting in changes in efficiency values.
4. Progress towards improved performance through simultaneous comparison of the AP method and the shortest path with parametric methods.

Practically, the proposed shortest path method can be used by practitioners as an alternative to the AP model, without concerns about infeasibility or discrepancies in input or output-oriented rankings. Additionally, when there is a need to adjust the importance of inputs and outputs, direct weighting can be easily applied.

In future research, a similar shortest path model can be designed with changes in objective function values to align with the definition of efficiency. Sensitivity analysis can be conducted to explore the impact of applied weights on obtained ranks. Furthermore, the relationship between the proximity of DMUs and inputs/outputs of the path can be examined. These methods can also be extended to non-deterministic cases, such as fuzzy, stochastic, and interval analysis. Simulation studies can explore the concept of popularity and compare the proposed method with other super-efficiency methods, considering both weighted and non-weighted models.

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